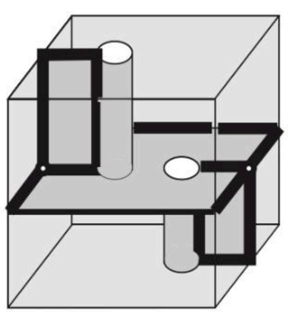
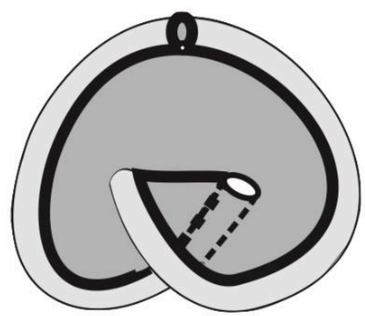
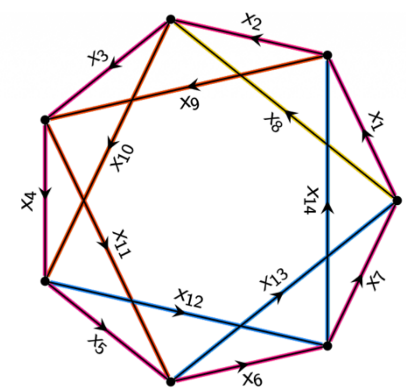
Non-semi-simple tensor categories and the stable Andrews-Curtis conjecture







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Algebraic structures

- Symmetries and local operators in quantum field theories, generalized symmetries, …

  e.g. vertex operator algebras and

  modular tensor categories from 2d CFTs
- Input data for quantum invariants,

  e.g. fusion categories for TQFTs
- …

Motivation

- Stable Andrews-Curtis conjecture (sACC)
- Exotic structure of 4-manifolds—4D SPC
- …

Need:

- Non-semisimple (NSS) tensor categories
- Do effective calculation with examples

Stable Andrews-Curtis Conjecture

Andrews–Curtis Conjecture, 1965

Every balanced presentation $\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$ of the trivial group can be transformed to the trivial presentation $\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$ by a sequence of Andrews–Curtis moves.

Andrews–Curtis moves:

- (AC1): $r_i \to r_i r_j$ for $j \neq i$
- (AC2): $r_i \to r_i^{-1}$
- (AC3): $r_i \to g r_i g^{-1}$ for $g \in \{x_1^{\pm 1}, \ldots, x_n^{\pm 1}\}$

Many potential counterexamples exist, but...
Bridson–Lishak: some AC-trivial presentations require superexponential path lengths

Stabilization: add a new generator x and relator x

sACC: any balanced presentation of the trivial group can be transformed to the empty presentation by AC moves + stabilization and inverses.

Akbulut-Kirby potential counterexamples

Akbulut–Kirby (1978)

$$\mathrm{AK}(n) = \langle x, y \mid xyx = yxy, x^n = y^{n+1} \rangle$$

- AK(1): AC-trivial (easy)
- AK(2): AC-trivial (Myasnikov, 1997)
- AK(3): Minimal-length potential counterexample (length 13)
- AK(n) for $n \geq 3$: Open!

We showed: arXiv:2408.15332

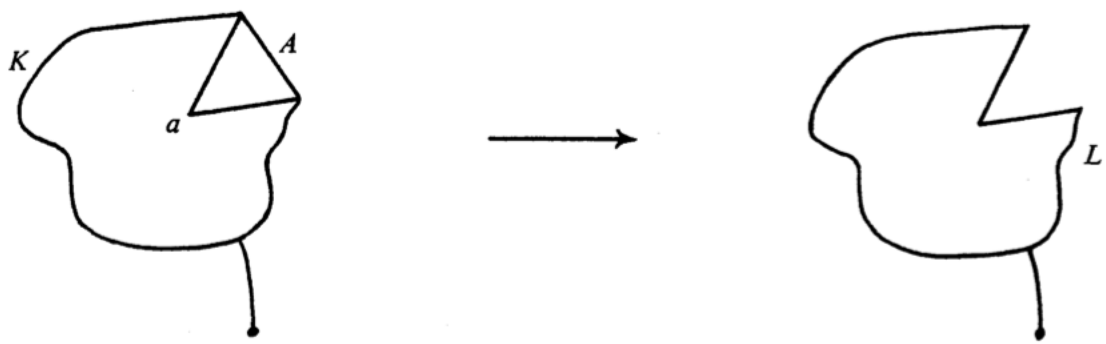
- AK(n) is AC-equivalent to a presentation of length $n + 11$
  - Reduction in length for $n \geq 5$
- Any trivialization of AK(3) must involve a presentation of length ≥ 27 (improves previous bound of 20)

Topological approach via fake surface

sACC=

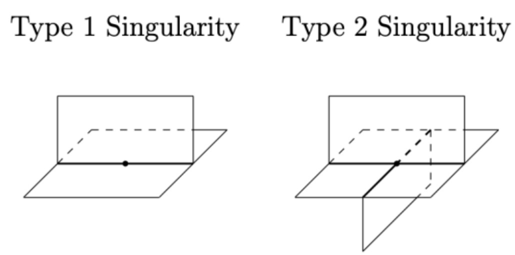
any two contractible cellular fake surfaces are **3-**deformable to each other or each is 3-deformable to a point.

n-deformation=collapses and expansions through n-dimensional balls with free faces: a combinatorial stronger version of homotopy

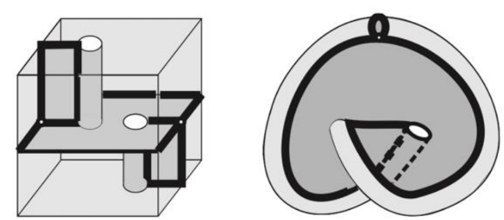


Cellular fake surface

- A singular 2-manifold with two kinds of singularities besides manifold points, and the intrinsic stratification is a CW complex (complexity=# of type 2 singularities):



- Examples:



Bing house with two rooms, Abalone

Matveev theorem

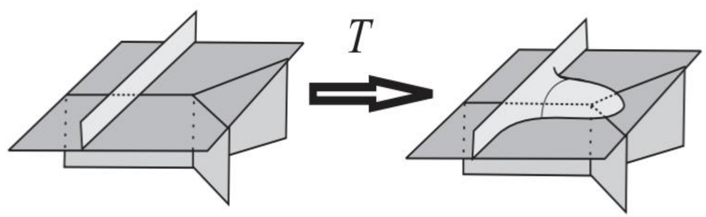
3-deformations of cellular fake surfaces are generated by two moves and their inverses: T,U and their inverses.

T move=dual of 2-3 Pachner move.

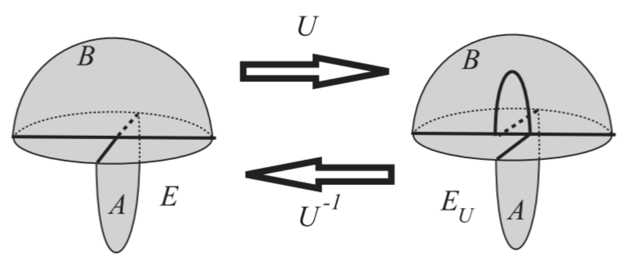
U is not realizable in R^3 and introduce a nontrivial T bundle.

Matveev moves

- T move:



- U move:



TQFT-like functors for counterexamples

- **Category of trivalent graphs and fake surfaces**

  objects: trivalent graphs

  morphisms: cellular fake surfaces with boundaries

  up to 3-deformations (T/U moves and inverses)

- **Goal:**

  the most general algebraic input, and

  produce counterexamples to sACC

Shadow topology

- Turaev defined quantum invariants of fake surfaces with gleams using ribbon fusion categories
- Not sensitive enough to detect counterexamples to sACC
- ??? NSS

Facile fusion categories

- Fusion category facile if self-dual, multiplicity free and all Frobenius-Schur indicators are 1—most interesting examples in TQC are all facile

  ---toric code, Ising, Fib
- Given by solutions of pentagons as F-matrices and F-symbols $F^{abc}_{d;mn}$ such that $F^{aaa}_{a;11} \neq 0$

How to generalize to FTCs

Finite tensor categories

- Representation categories of finite dimensional weak quasi-Hopf algebras
- Axioms:

A finite tensor category (FTC) is an abelian monoidal category $\mathcal{C}$ such that

- All hom spaces are finite-dimensional vector spaces and composition and $\otimes$ are $\mathbb{C}$-linear
- $\mathcal{C}$ is rigid (i.e. every object has left/right duals)
- $\mathrm{End}_{\mathcal{C}}(1) \simeq \mathbb{C}$
- $\mathcal{C}$ has enough projectives
- Every object has finite length (Jordan-Hölder series)
- Finitely many simple objects up to isomorphism

Representation categories of Nicholas Hopf algebras

Nichols Hopf algebras, K_n are some of the simplest Hopf algebras whose representation categories are non-semisimple. The 2^{n+1}-dimensional algebras are generated as an algebra by $n+1$ elements $K, \xi_1, \xi_2, \dots, \xi_n$ subject to the relations

$$\begin{aligned} K^2 = 1 \quad \xi_i^2 = 0 \quad K\xi_i = -\xi_i K \quad \xi_i\xi_j = -\xi_j\xi_i \\ \epsilon(K) = 1 \quad \epsilon(\xi_i) = 0 \quad S(K) = K \quad S(\xi_i) = -K\xi_i \\ \Delta(K) = K \otimes K \quad \Delta(\xi_i) = K \otimes \xi_i + \xi_i \otimes 1 \end{aligned}$$

Even with a relatively simple set up, even when $n = 2$, there are already uncountably infinitely many finite-dimensional indecomposable modules.

Exterior algebras extended by Z_2

K_2 is 8 dim and very interesting—3D SW theory

Nicholas Hopf algebras not Nicholas algebras

Green ring of Hopf algebra

Definition: Green Ring

As a set, the Green ring of a algebra $R(A)$ is the $\mathbb{Z}$ span of isomorphism classes of finite-dimensional indecomposable A-modules with addition given by $\oplus$ and multiplication by $\otimes$.

Using the Green ring of DK_1 (Chen 2014), we can calculate the Green ring of K_2. Let $X = \{g, x, y, z, X'_{n,\eta} | n \geq 1, \eta \in \mathbb{C}P^1\}$. Let J be the ideal of $\mathbb{Z}[X]$ generated by

$$g^2 - 1,\ x^3 - 2x(1+g),\ x(y-1-2g),\ x(y-z),\ yz - 1 - 2x^2,\ xX'_{n,\eta} - n(1+g)x$$
$$yX'_{n,\eta} - ngx^2 - gX'_{n,\eta},\ zX'_{n,\eta} - nx^2 - gX'_{n,\eta},\ X'_{n,\eta}X'_{s,\alpha} - nsgx^2$$
$$X'_{n,\eta}X'_{t,\eta} - n(t-1)gx^2 - X'_{n,\eta} - gX'_{n,\eta}$$

where $n, s, t \geq 1$ with $t \geq n$, $\eta, \alpha \in \mathbb{C}P^1$ with $\eta \neq \alpha$.
Let $\tilde{X} = \{g^2, x, y, z, X'_{n,\eta} | n \geq 1, \eta \in \mathbb{C}P^1\}$.

Theorem: Green Ring of K_2 (J-Wang 2025)

$R(K_2) \simeq \mathbb{Z}[\tilde{X}]/J$

Finite reconstruction of FTCs

- A linear category is a system of algebras Hom(x,x) and bimodules Hom(x,y)
- Objectless presentation with only morphisms

Projective ideal as complete invariant

Projective ideal:
the non-unital tensor category of all finitely dimensional projective modules of a weak quasi-Hopf algebra

Theorem: The Projective Ideal is a Complete Invariant (J-Wang 2025)

Let $\mathcal{C}$ be an abelian category with any additional structures and let $\mathcal{P}$ be its full projective ideal. Then $\mathcal{C}$ is the unique abelian completion of $\mathcal{P}$ up to the corresponding natural equivalence.

Any object A is identified with the image of the composition of the projective cover and injective hull morphisms ϕ_A. A morphism $f : A \to B$ can be recovered as a morphism from the projective cover of A to the injective hull of B that descends to a morphism $\operatorname{im}\phi_A \to \operatorname{im}\phi_B$.

Projective Auslander algebra

Definition: Projective Auslander Algebra

For a finite tensor category $\mathcal{C}$ with projective indecomposables $\{P_i\}_{i=0}^{n}$, then let P denote the projective generator $\bigoplus_{i=0}^{n} P_i$. We call $\mathrm{End}_{\mathcal{C}}(P)$ the projective Auslander algebra (PAA) of $\mathcal{C}$.

Since P is a projective generator and the category has enough projectives/injectives, every object A can be viewed as the image of a morphism $nP \to mP$. This also implies that the associators of any object can be recovered from $a_{P,P,P}$.

Idea:

Work with PAA modules of morphism spaces, and intertwiners of modules instead of vector spaces, so F matrices are module maps.

All $A^{\otimes n} - A^{\otimes m}$ bimodules called A-module

NSS vs SS

Let $\{V_i\}_{i=0}^m$ denote the simple objects (up to isomorphism) and $\{P_i\}_{i=0}^m$ the corresponding projective indecomposables.

Projective Auslander Algebra of K_n

- For a finite-dimensional Hopf algebra H, $H \simeq \bigoplus_i \dim(V_i) P_i$.
- So if $\dim(V_i) = 1$ as is the case for K_n, then $H \simeq \bigoplus_i P_i = P$.
- Therefore, $A \simeq \mathrm{End}_{\mathcal{C}}(P) \simeq \mathrm{Hom}_{K_n}(K_n, K_n) \simeq K_n$ as an algebra.

Semisimple Case

- Every object is projective, $P \simeq \bigoplus_i V_i$.
- Since $\mathrm{Hom}_{\mathcal{C}}(V_i, V_j) \simeq \delta_{i=j}\mathbb{C}$, $A = \mathrm{End}_{\mathcal{C}}(P) \simeq \mathbb{C}^{m+1}$.
- A only detects the number of simple objects.

Therefore, the projective Auslander algebra contains much more information in the non-semisimple case.

One pentagon rules all

Suppose P is a projective generator, $a_{P,P,P}$ is a morphism $(P \otimes P) \otimes P \to P \otimes (P \otimes P)$ such that $a_{P,P,P} \circ (f \otimes g) \otimes h = f \otimes (g \otimes h) \circ a_{P,P,P}$ for any morphisms $f, g, h : P \to P$. Suppose that the pentagon commutes when all objects are P using the induced associator on tensor products of P. Then, the pentagons commute for any objects of the category.

$$
\begin{tikzcd}
 & & \bigoplus((P \otimes P) \otimes P) \otimes P \arrow[dll] \arrow[d] \arrow[drr] & & \\
\bigoplus(P \otimes (P \otimes P)) \otimes P \arrow[dr] \arrow[ddd] & & ((A_1 \otimes A_2) \otimes A_3) \otimes A_4 \arrow[dl] \arrow[dr] & & \bigoplus(P \otimes P) \otimes (P \otimes P) \arrow[dl] \arrow[ddd] \\
 & (A_1 \otimes (A_2 \otimes A_3)) \otimes A_4 \arrow[d] & & (A_1 \otimes A_2) \otimes (A_3 \otimes A_4) \arrow[d] & \\
 & A_1 \otimes ((A_2 \otimes A_3) \otimes A_4) \arrow[rr] & & A_1 \otimes (A_2 \otimes (A_3 \otimes A_4)) & \\
\bigoplus P \otimes ((P \otimes P) \otimes P) \arrow[ur] \arrow[rrrr] & & & & \bigoplus P \otimes (P \otimes (P \otimes P)) \arrow[ul]
\end{tikzcd}
$$

with M. Jubeir

Tensor system for FTC

Let $\mathcal{C}$ be a locally-finite monoidal category with enough projectives such that $\otimes$ is bilinear on morphisms. Let $A = \bigoplus_{i,j} \hom(P_i, P_j)$ and let $\psi \in A$ denote the map $P \to 1 \to P$ representing a choice of projective cover and injective hull of the tensor unit. Then, suppose $P_i \otimes P_j \simeq \bigoplus N_{i,j}^k P_k$. Then, this gives $\bigoplus_l \hom(P_l, \bigoplus_k N_{i,j}^k P_k)$ a left A module structure, Let $T = \bigoplus_{i,j,k} \hom(P_i \otimes P_j, N_{i,j}^k P_k)$ can be viewed as an $A - A \otimes A$ bimodule. Let $F = \bigotimes_{i,j,k,l} F_l^{i,j,k}$ be the map $F_l^{i,j,k} : \hom(P_i \otimes P_j, \bigoplus N_{i,j}^m P_m) \otimes \hom(P_m \otimes P_k, P_l) \to \hom(P_i \otimes P_m, P_l) \otimes \hom(P_j \otimes P_k, \bigoplus N_{j,k}^m)$ be the map induced by the associator and let $l = \bigoplus l_i, r = \bigoplus r_i$ denote the unit morphisms in $\hom(P_i \otimes P, P_i)$ and $\hom(P \otimes P_i, P_i)$ where we act by ψ on P. Then, $\mathcal{C}(A, \psi, T, F, l, r)$ is a monoidal system.

Conversely, given a monoidal system, then a finite tensor category can be constructed. Furthermore, $\mathcal{C}$ is equivalent to the category $\mathcal{C}(A, \psi, T, F, l, r)$.

Let $\mathcal{C}$ be a finite tensor category. Then a tensor system $\mathcal{T} = \{A, \psi, T, F, l, r, S, \delta, \epsilon\}$ arose from the category. Furthermore, any such tensor system yields a fintie tensor category $\mathcal{C}(\mathcal{T})$. Two such tensor categories are equivalent if and only if they differ by a gauge transformation.

with M. Jubeir

Algebraicalization of U move

- ??? most general solutions to a state sum invariant of cellular fake surfaces up to T, U moves and their inverses?
- Disprove sACC, e.g. show AK(3) is a counterexample

Classification of low rank FTCs

- Total rank=# of simples and indecomposable projectives
- Total rank of NSS FTCs ≥ 4
- Total rank of NSS modular categories ≥ 5 with at least one Steinberg (both simple and projective)
- Partial results for PAA has a Hopf algebra structure

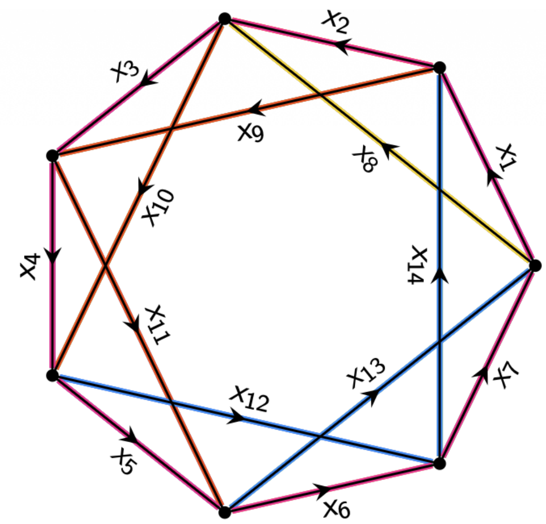
Could sACC hold?

- Fake surface approach:

  embedded disk conjecture and induction

  --- every contractible cellular fake surface

  has an embedded disk
- AI and ML to trivialize AK(3):



Zeeman conjecture

For every contractible 2-complex K,
K x I is collapsable, I=[0,1].

1. K=cellular fake surface (CFS) spines of the 3-ball,

   ZC=3D Poincare conjecture
2. K=non-spine CFSs, ZC=sACC