Formal Concept Analysis

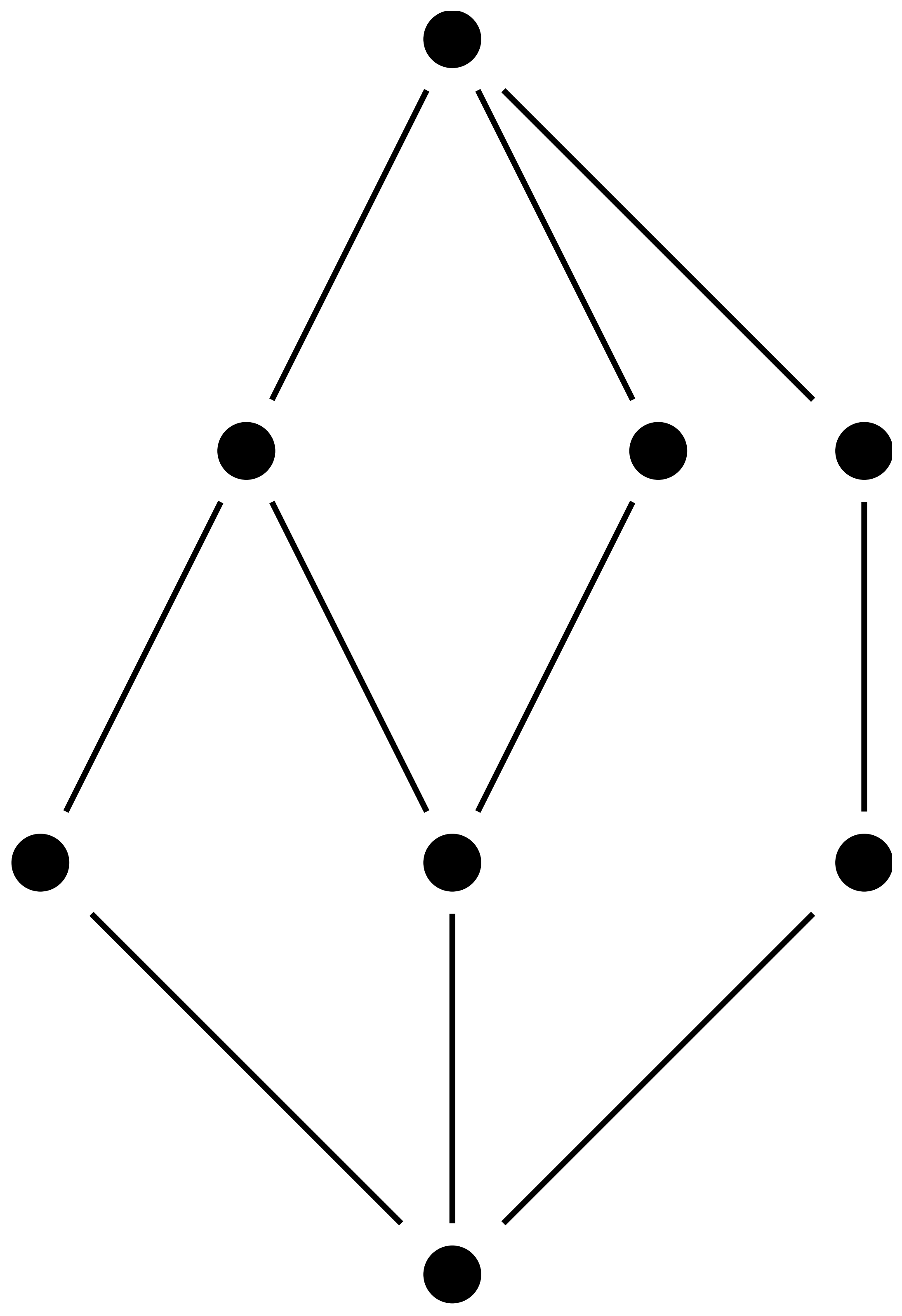
and Homotopical Combinatorics

and You

Ben Spitz - Algebraic Structures in Topology, July 16 2026 ———————————————————— joint work with Scott Balchin

Prologue

Finite Lattices



This is a finite lattice

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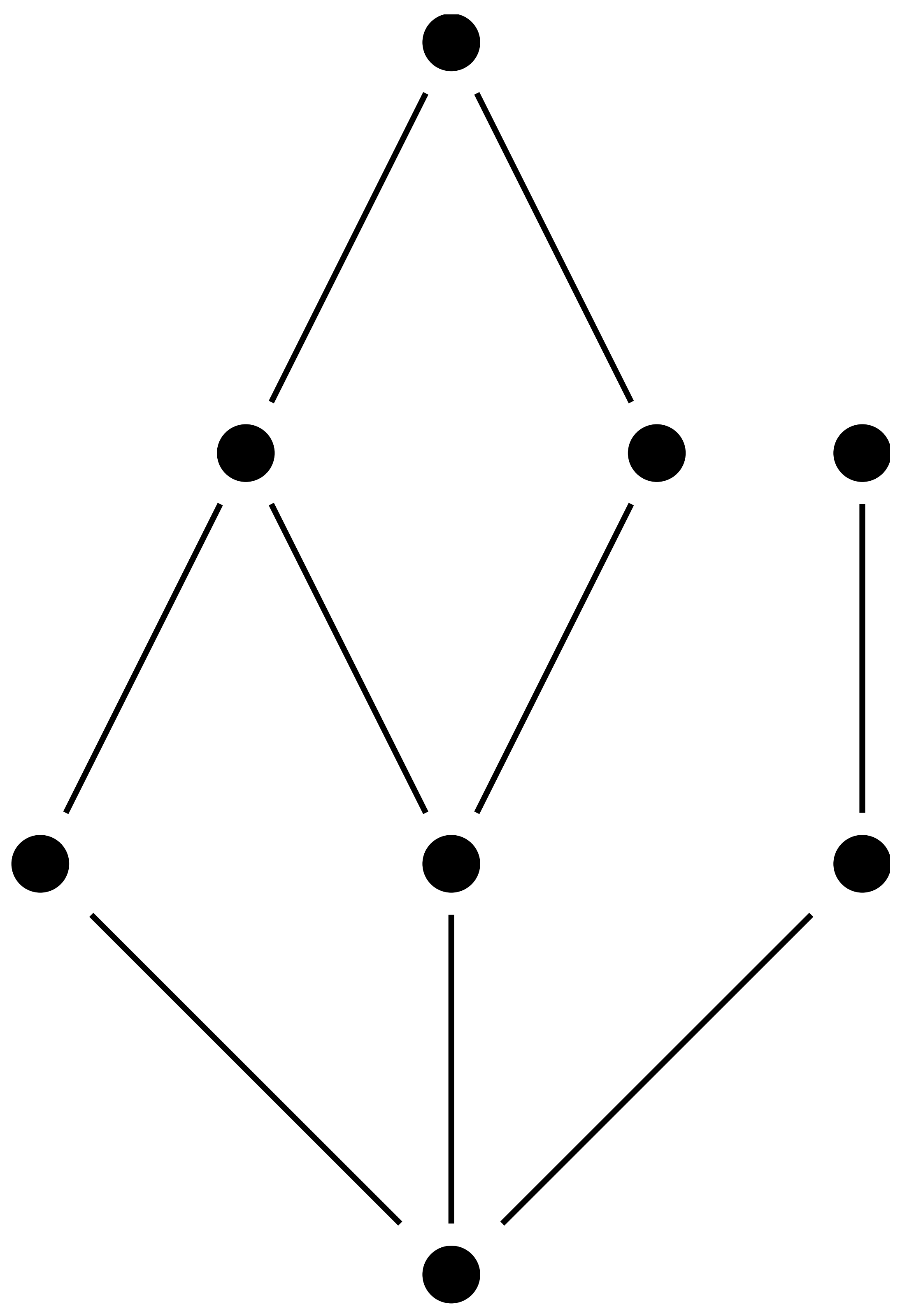
Finite Lattices



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Prologue

Finite Lattices

Question How many bits of data does it take to describe a finite lattice?

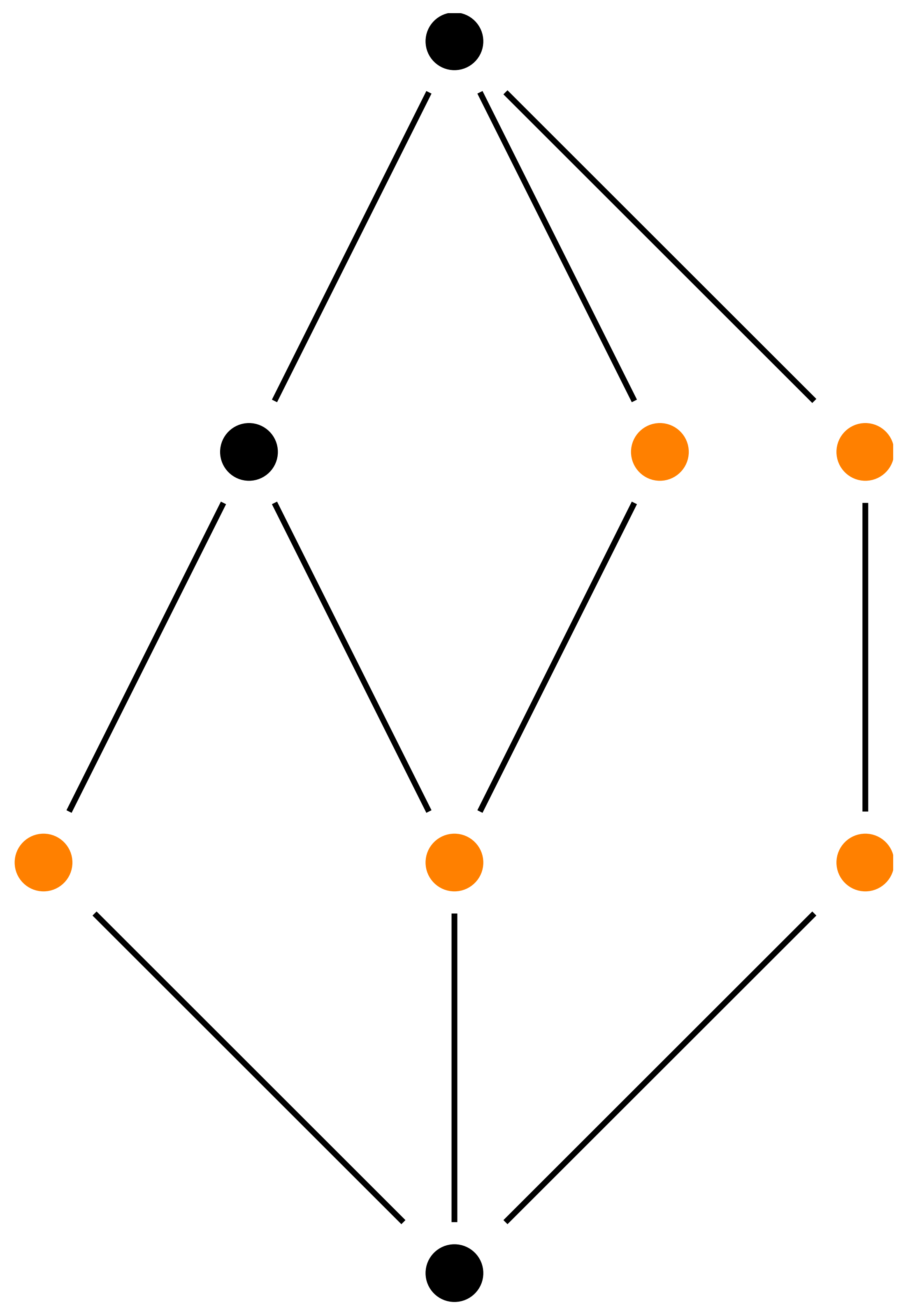
Chapter 1

The Fundamental Theorem of FCA

An element x is said to be *join-irreducible* if

(a) It is not the bottom element of the lattice;

(b) It is not the join of two other elements.



Chapter 1

The Fundamental Theorem of FCA

An element x is said to be *meet-irreducible* if

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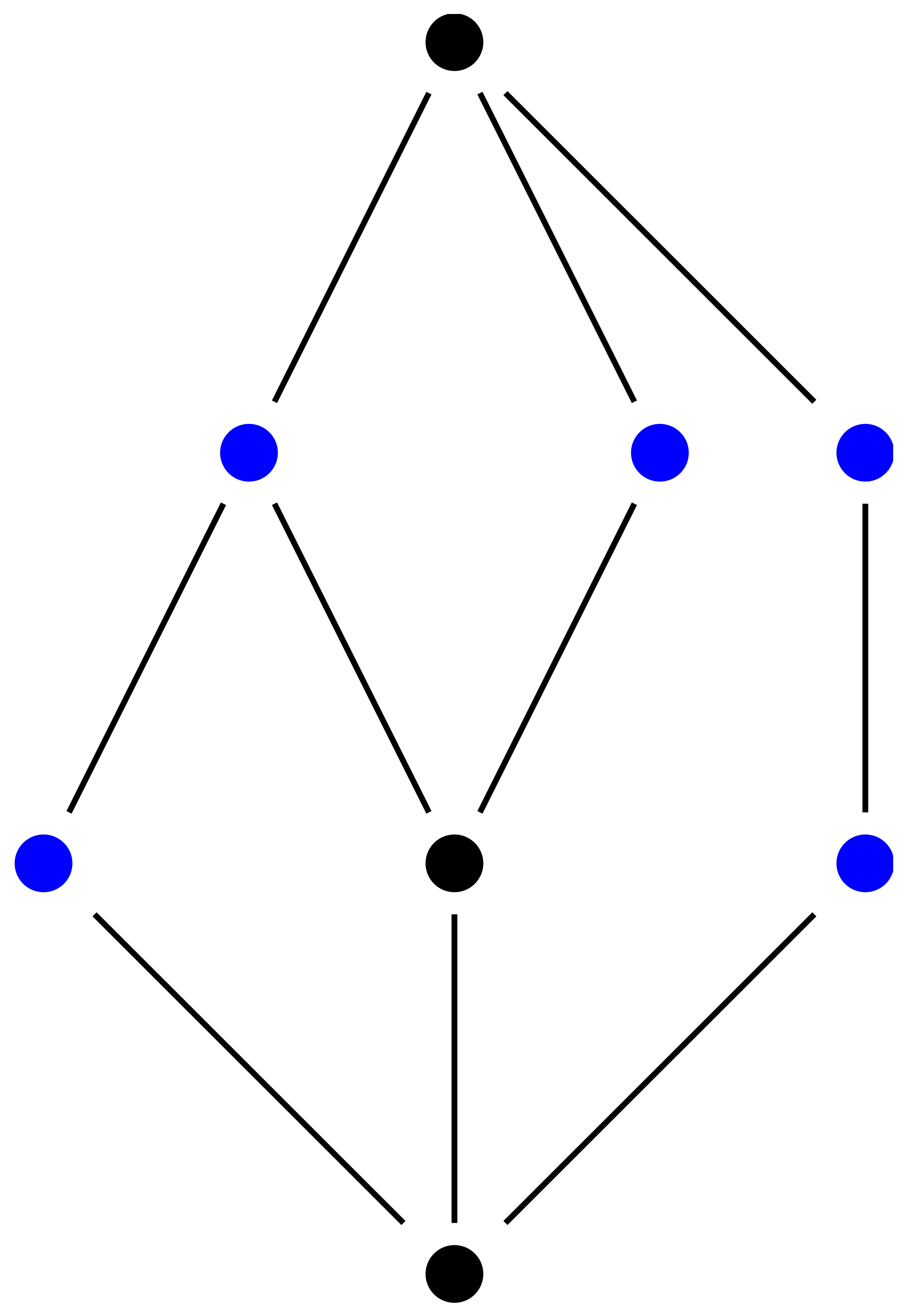

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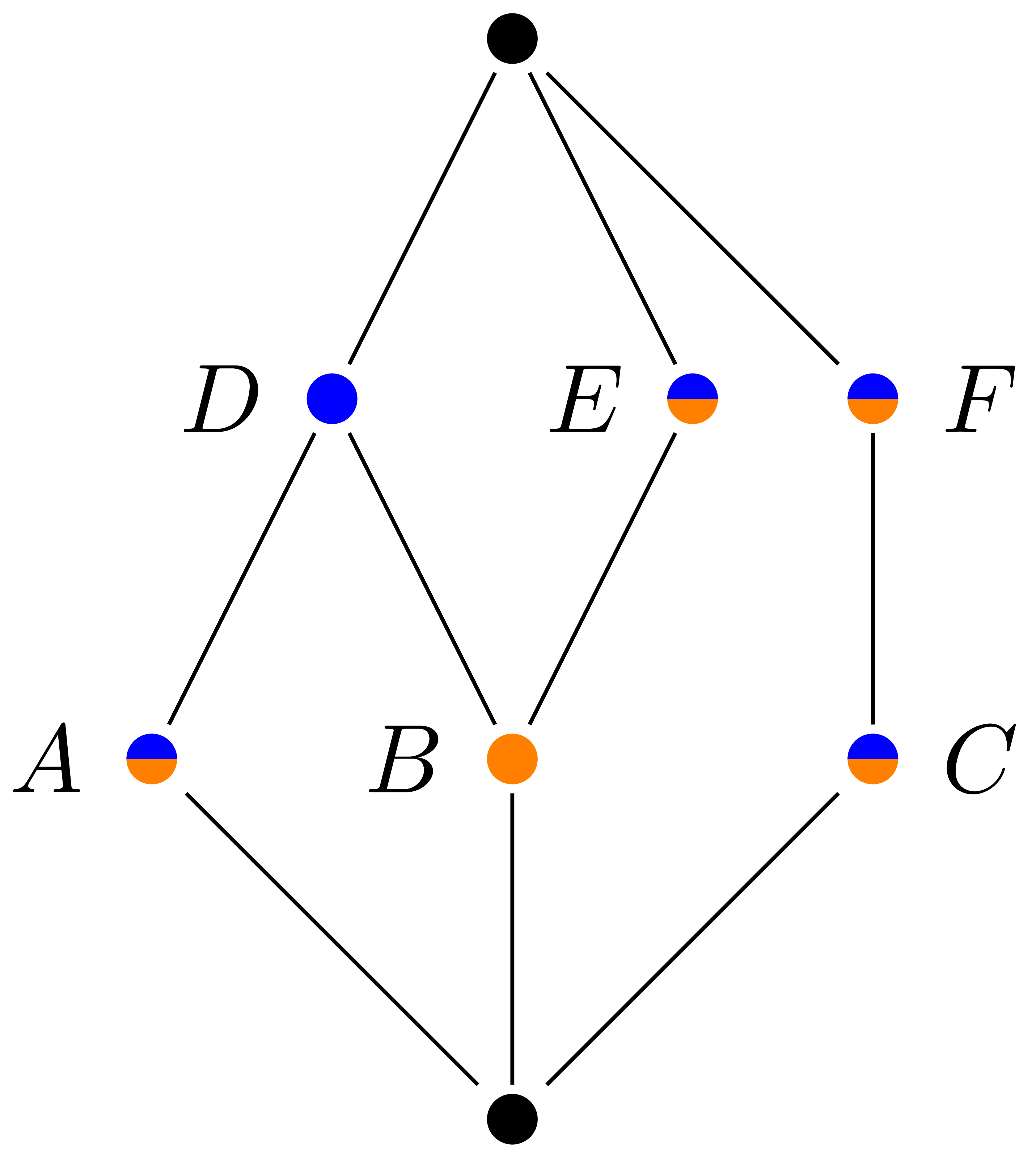
The Fundamental Theorem of FCA

$J(L) = \{A, B, C, E, F\}$ $\quad M(L) = \{A, C, D, E, F\}$



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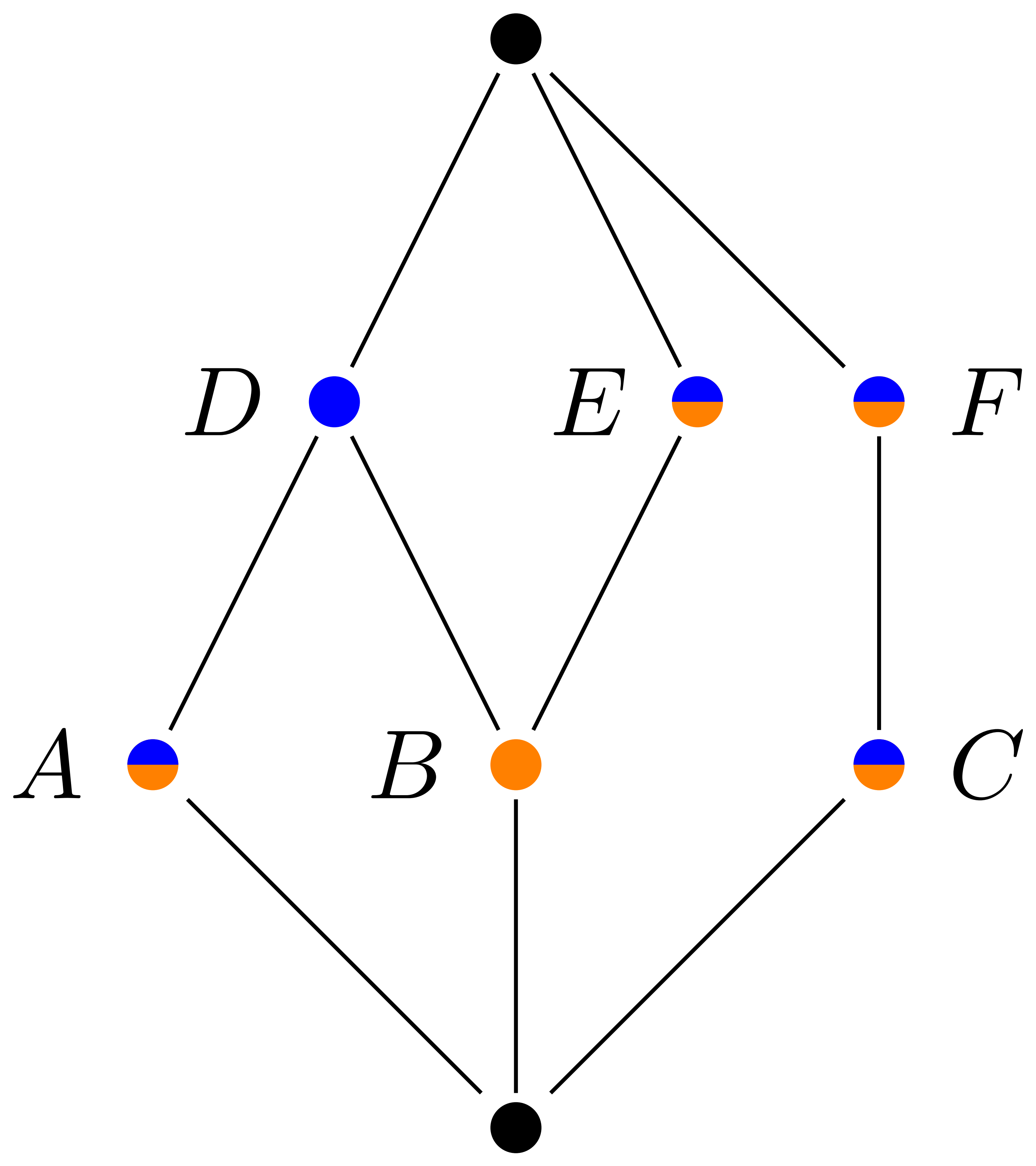

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The Fundamental Theorem of FCA

Theorem The original lattice can be recovered (up to iso.) from this matrix!

$$\begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

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Idea: every element in the lattice is both
(i) a meet of meet-irreducibles
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Idea: every element in the lattice is both
(i) a meet of meet-irreducibles
(ii) a join of join-irreducibles

So, we encode each element as a pair (X, Y), where $X \subseteq J(L)$ and $Y \subseteq M(L)$.

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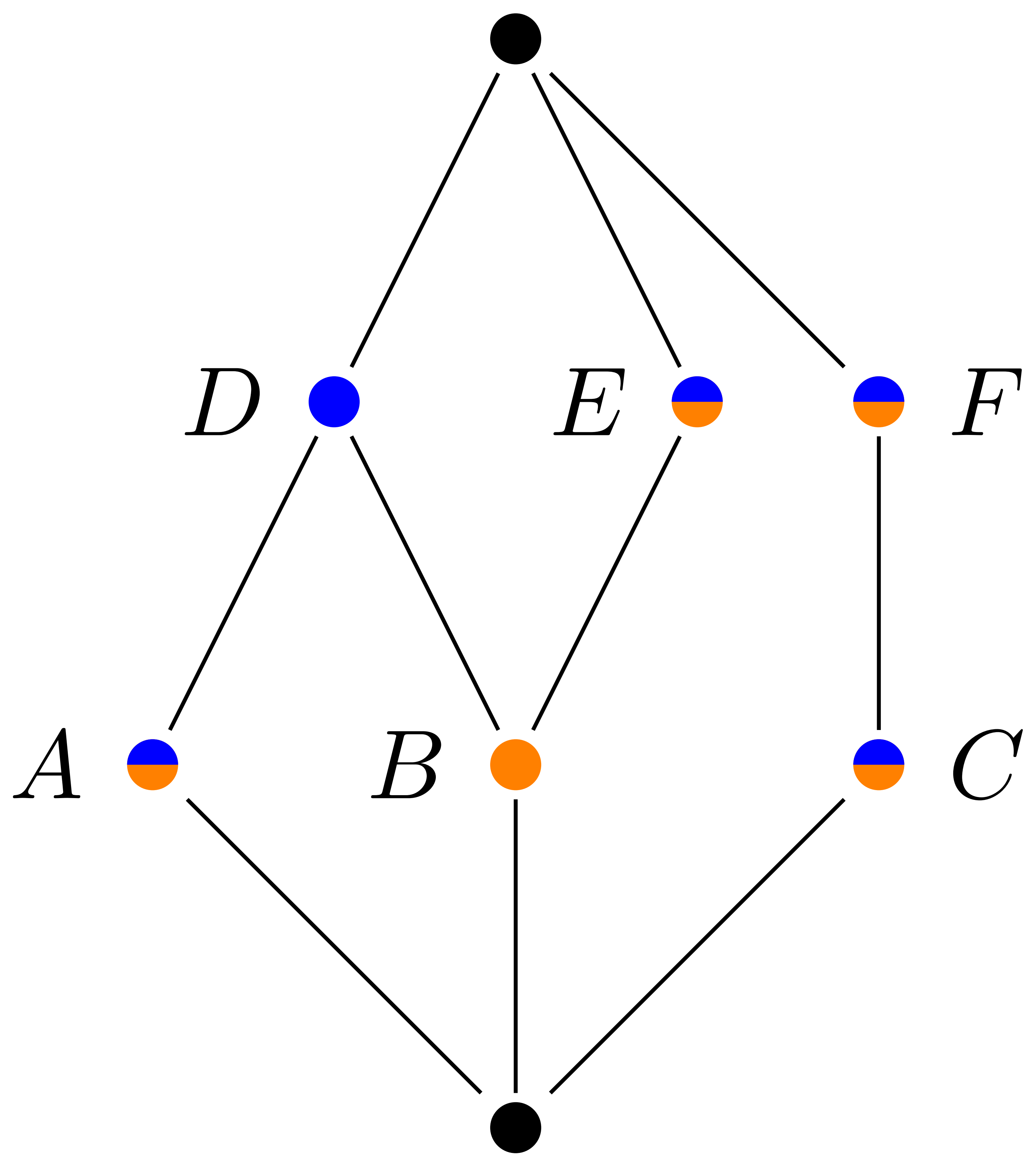

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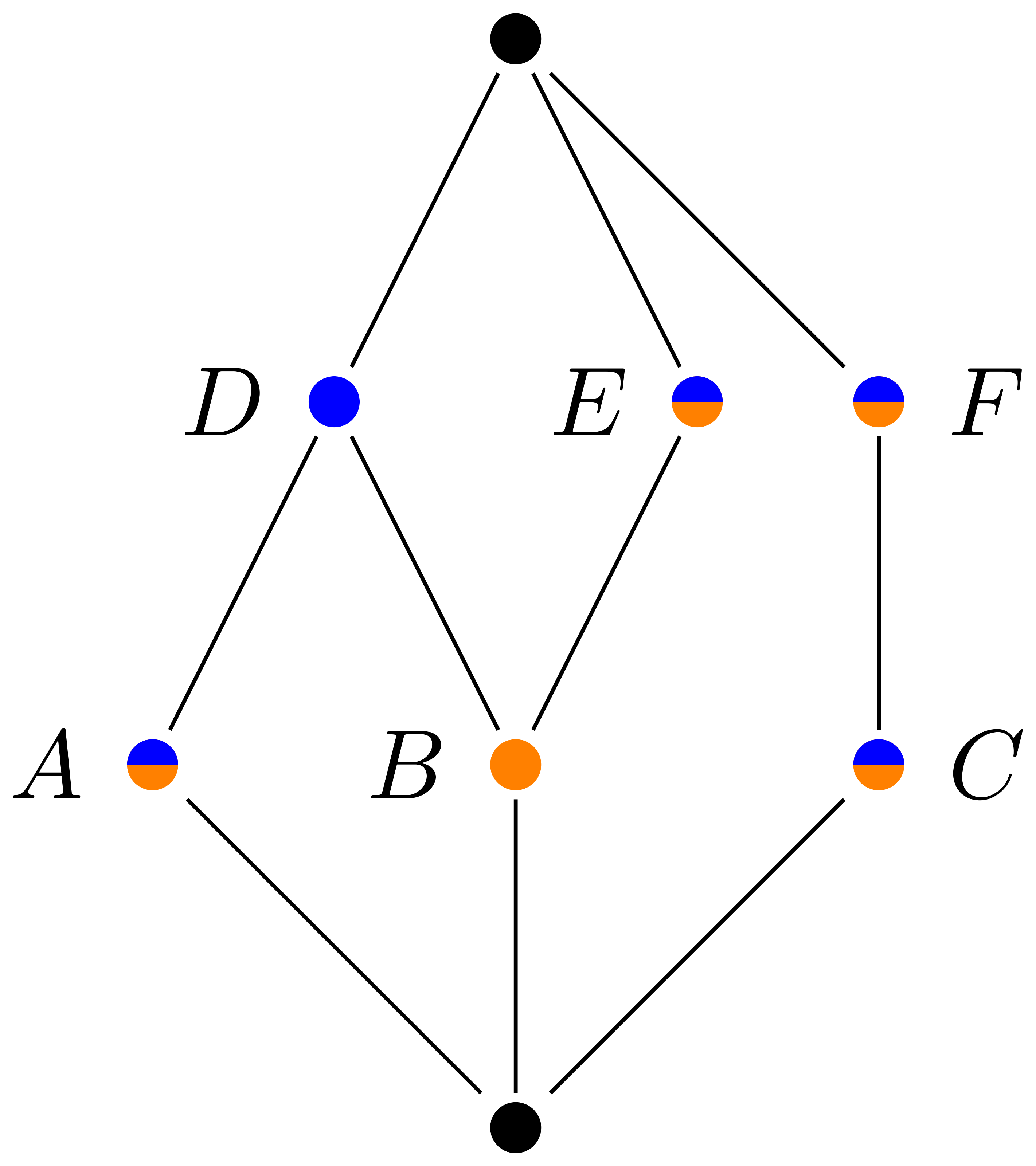
$J(L) = \{A, B, C, E, F\}$ $\quad$ $M(L) = \{A, C, D, E, F\}$

We encode each element $\ell \in L$ as the pair (X, Y) where

$$X = \{j \in J(L) : j \leq \ell\}$$
$$Y = \{m \in M(L) : \ell \leq m\}$$



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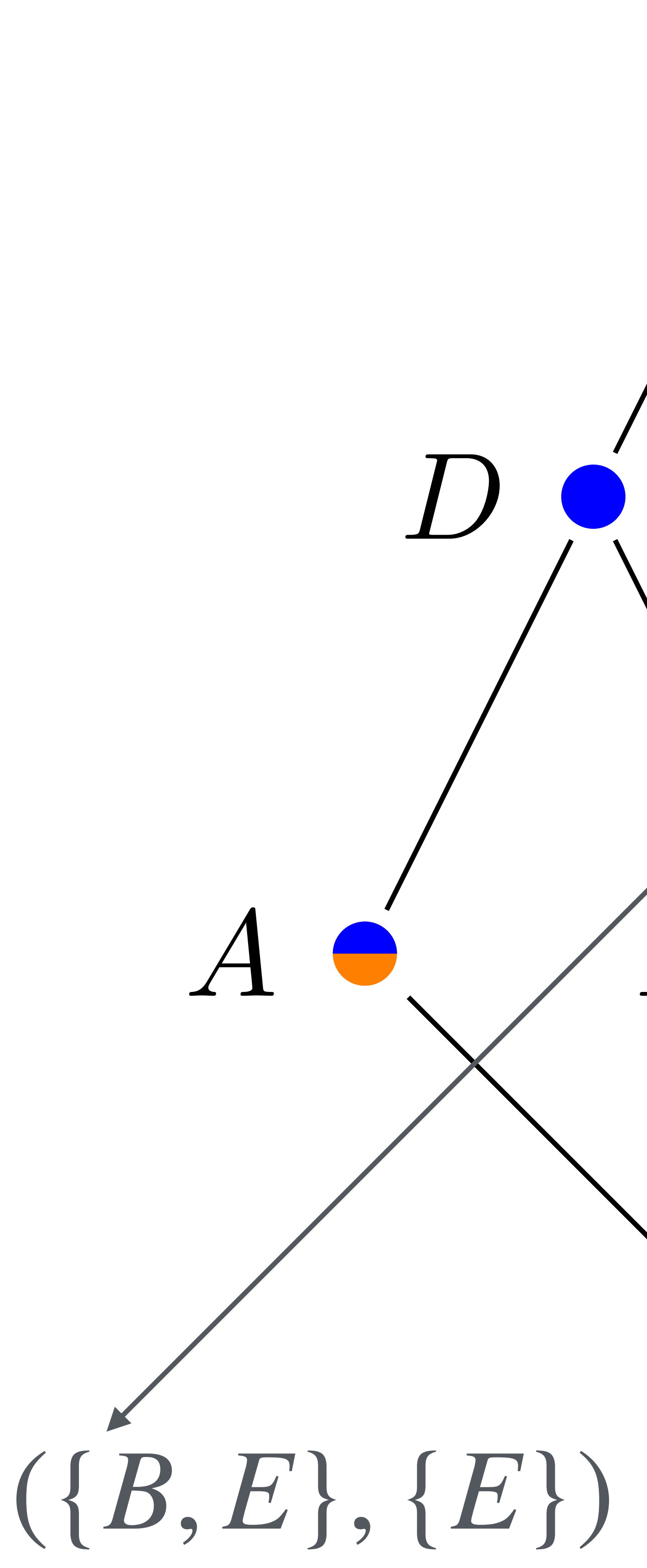

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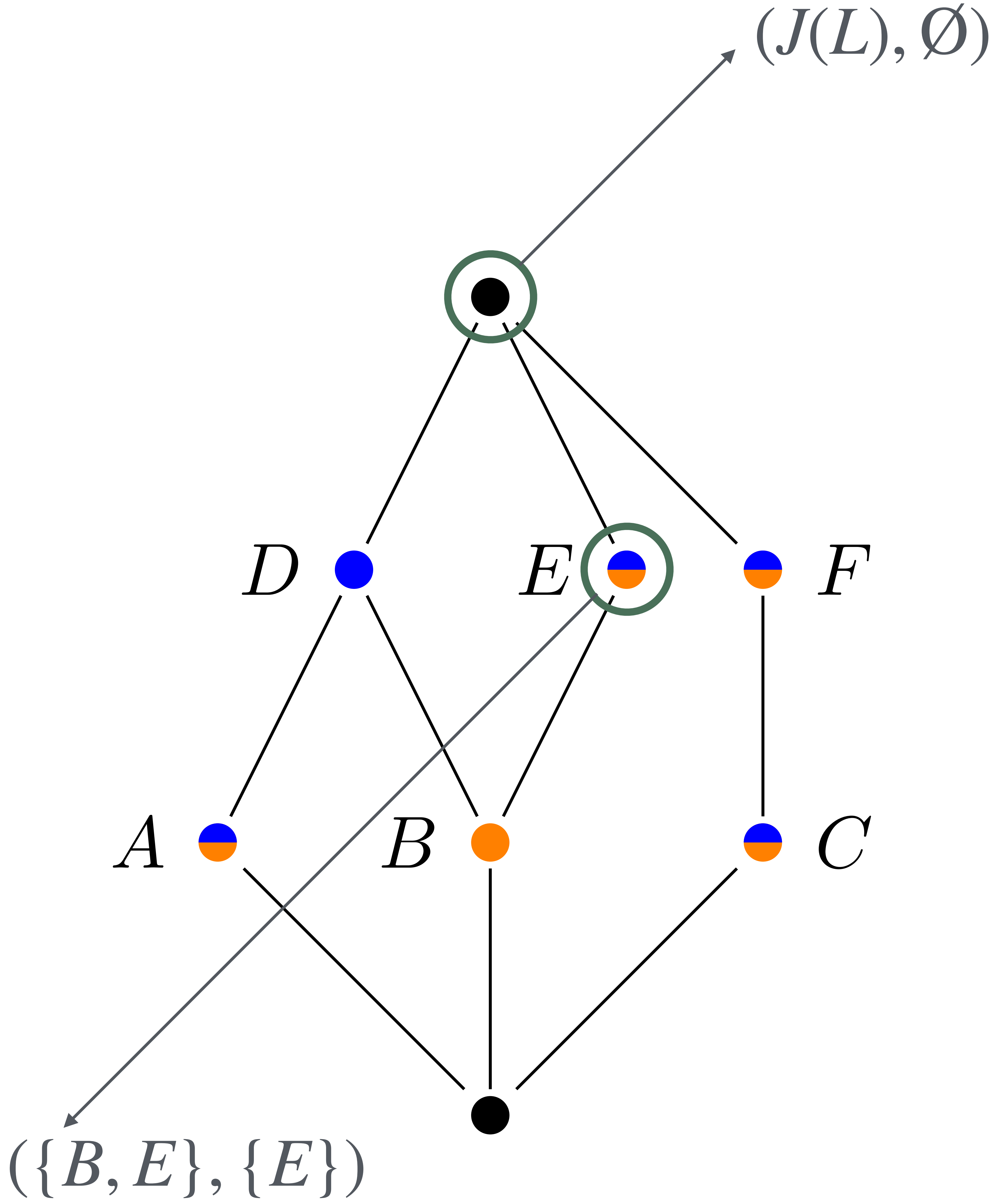

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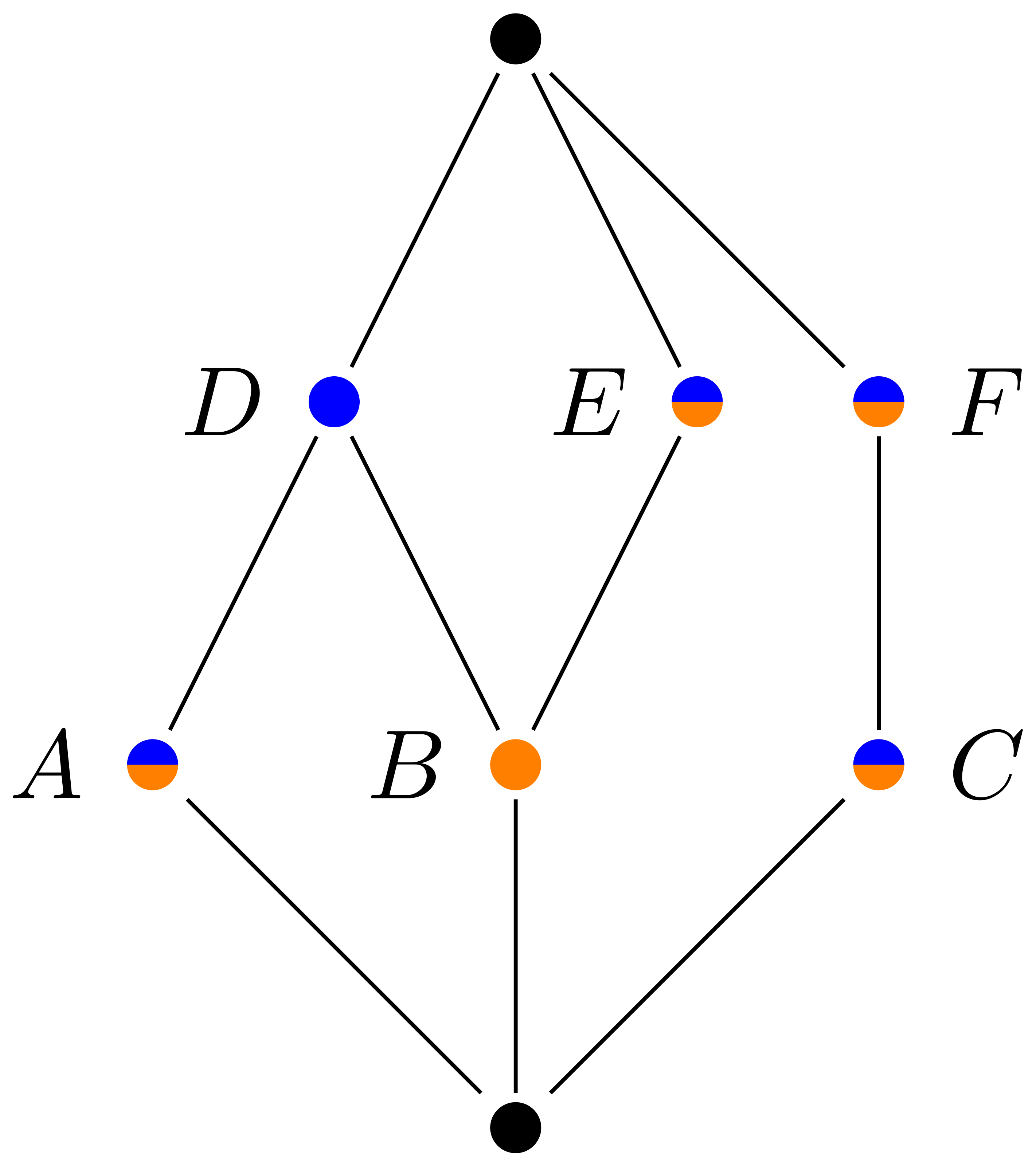
Each such pair satisfies

$$X = Y^{\downarrow} := \{j \in J(L) : \forall y \in Y(j \leq y)\}$$

and

$$Y = X^{\uparrow} := \{m \in M(L) : \forall x \in X(x \leq m)\}$$



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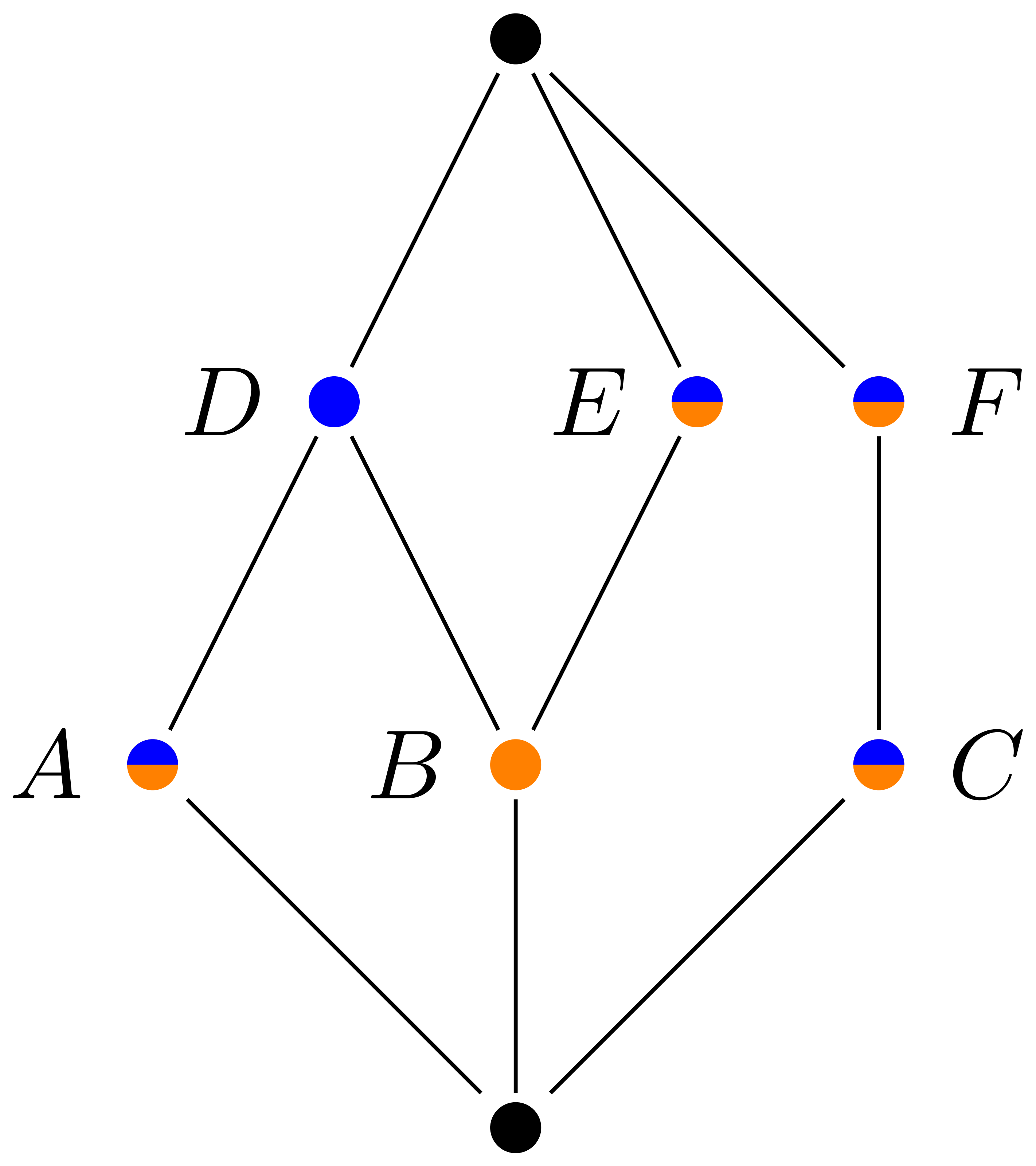
We encode each element $\ell \in L$ as the pair (X, Y) where $X = \{j \in J(L) : j \leq \ell\}$ and $Y = \{m \in M(L) : \ell \leq m\}$.

This defines a bijection from L to the set of pairs

$$\{(X, Y) \in 2^{J(L)} \times 2^{M(L)} : X = Y^{\downarrow}, Y = X^{\uparrow}\}$$

!!!!



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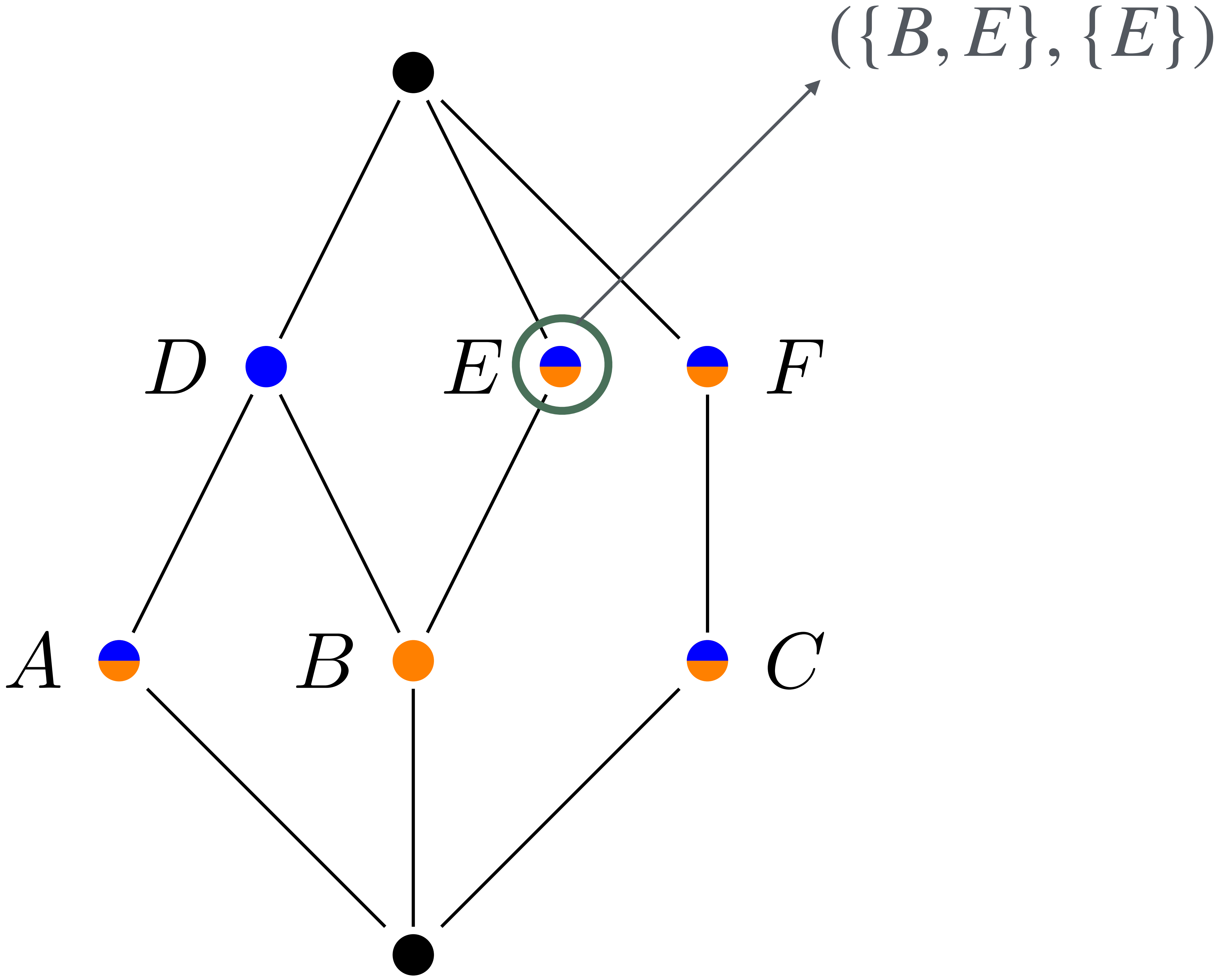

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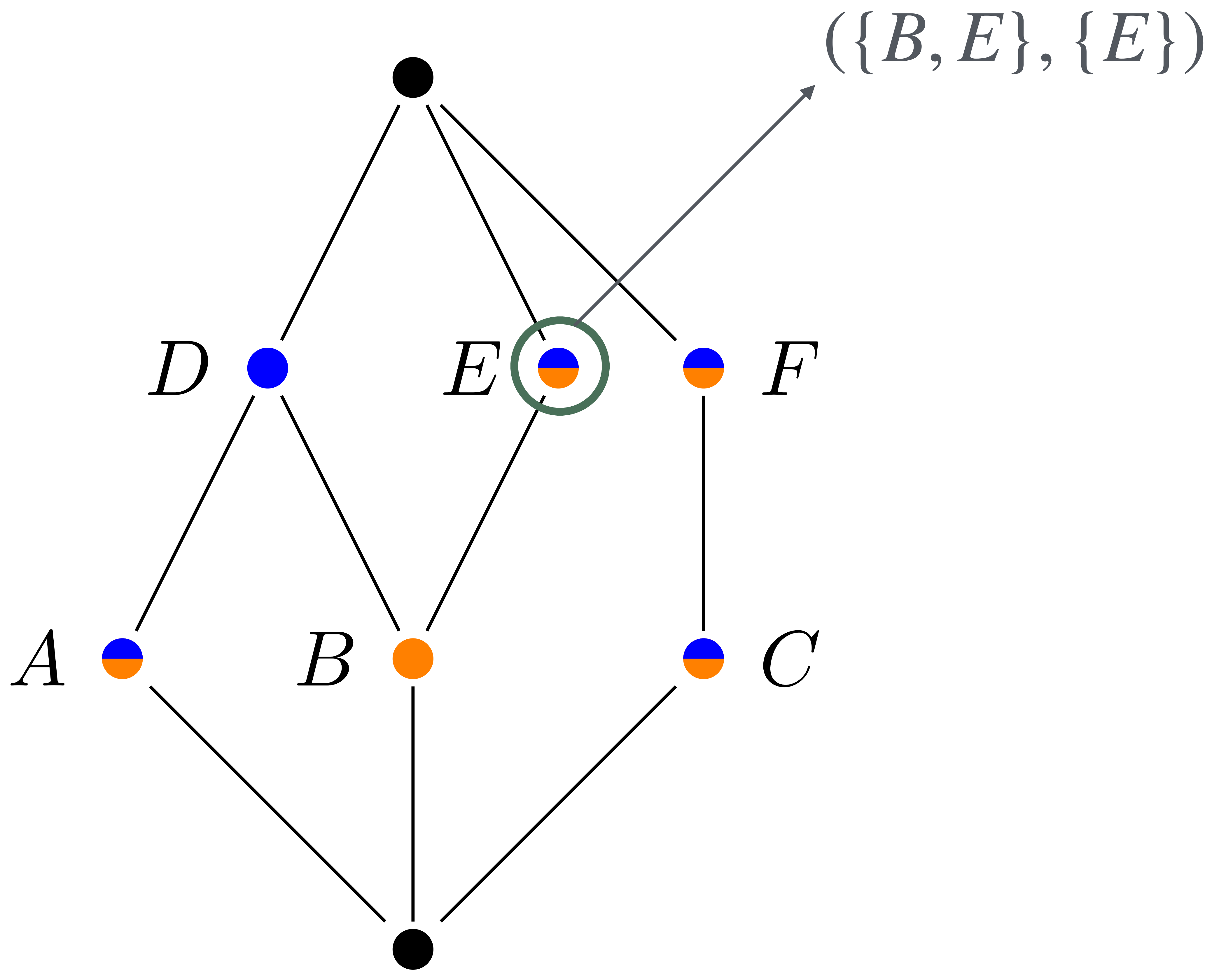

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$$\begin{array}{ccccc}
1 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
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\end{array}$$

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$$\begin{matrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{matrix}$$

Elements of L correspond bijectively with "maximal submatrices of 1's"

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Elements of L correspond bijectively with "maximal submatrices of 1's"

There exist algorithms for efficiently enumerating these submatrices!

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3. Plug this matrix into some FCA software

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4. Profit!

Interlude

Living in Water

	needs water to live	lives in water	lives on land	needs chlorophyll	dicotyledon	monocotyledon	can move	has limbs	breast feeds
fish leech	×	×					×		
bream	×	×					×	×	
frog	×	×	×				×	×	
dog	×		×				×	×	×
water weeds	×	×		×		×			
reed	×	×	×	×		×			
bean	×		×	×	×				
corn	×		×	×		×			

Image from Uta Priss — https://upriss.github.io/fca/examples.html

Interlude

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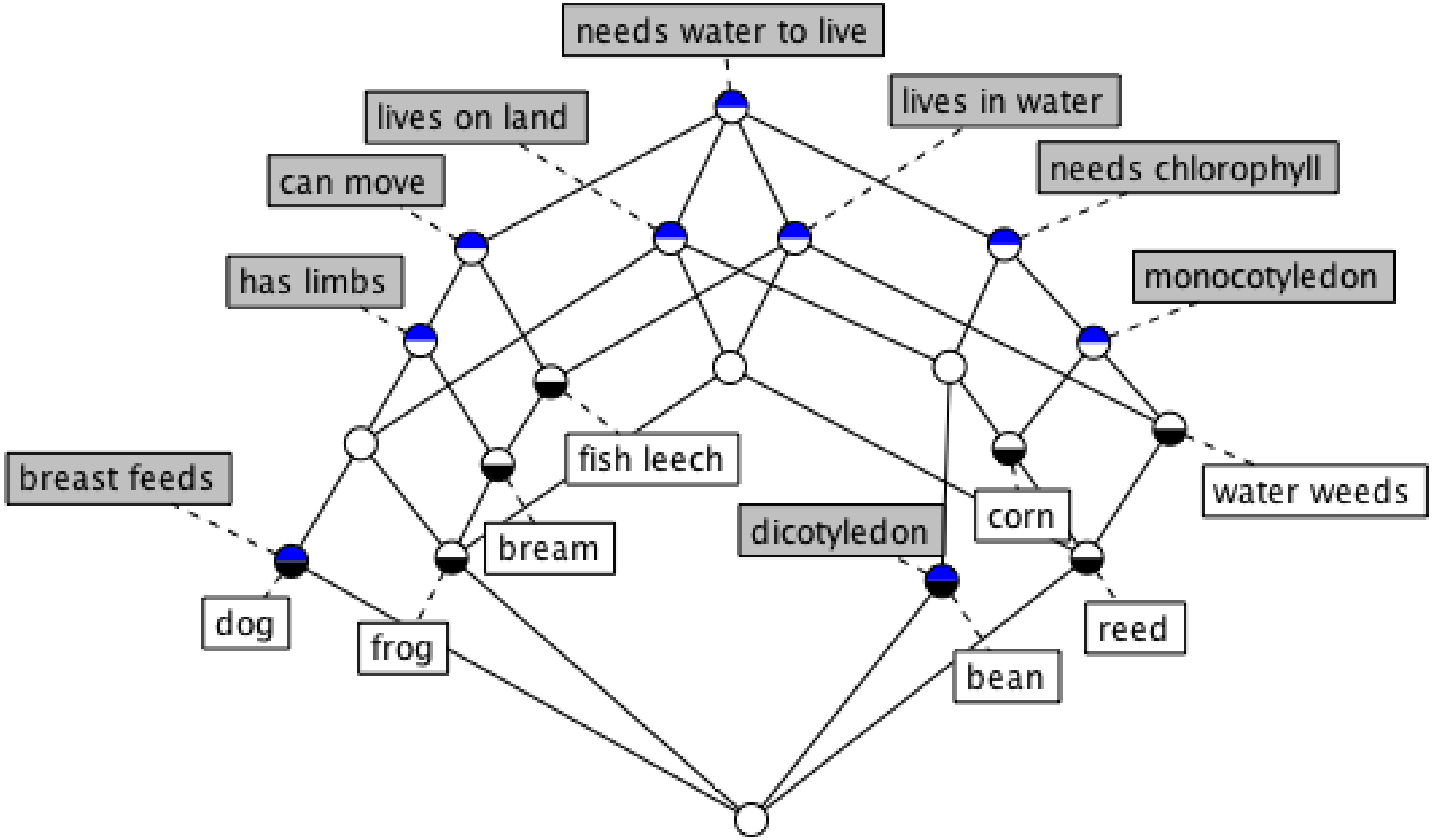

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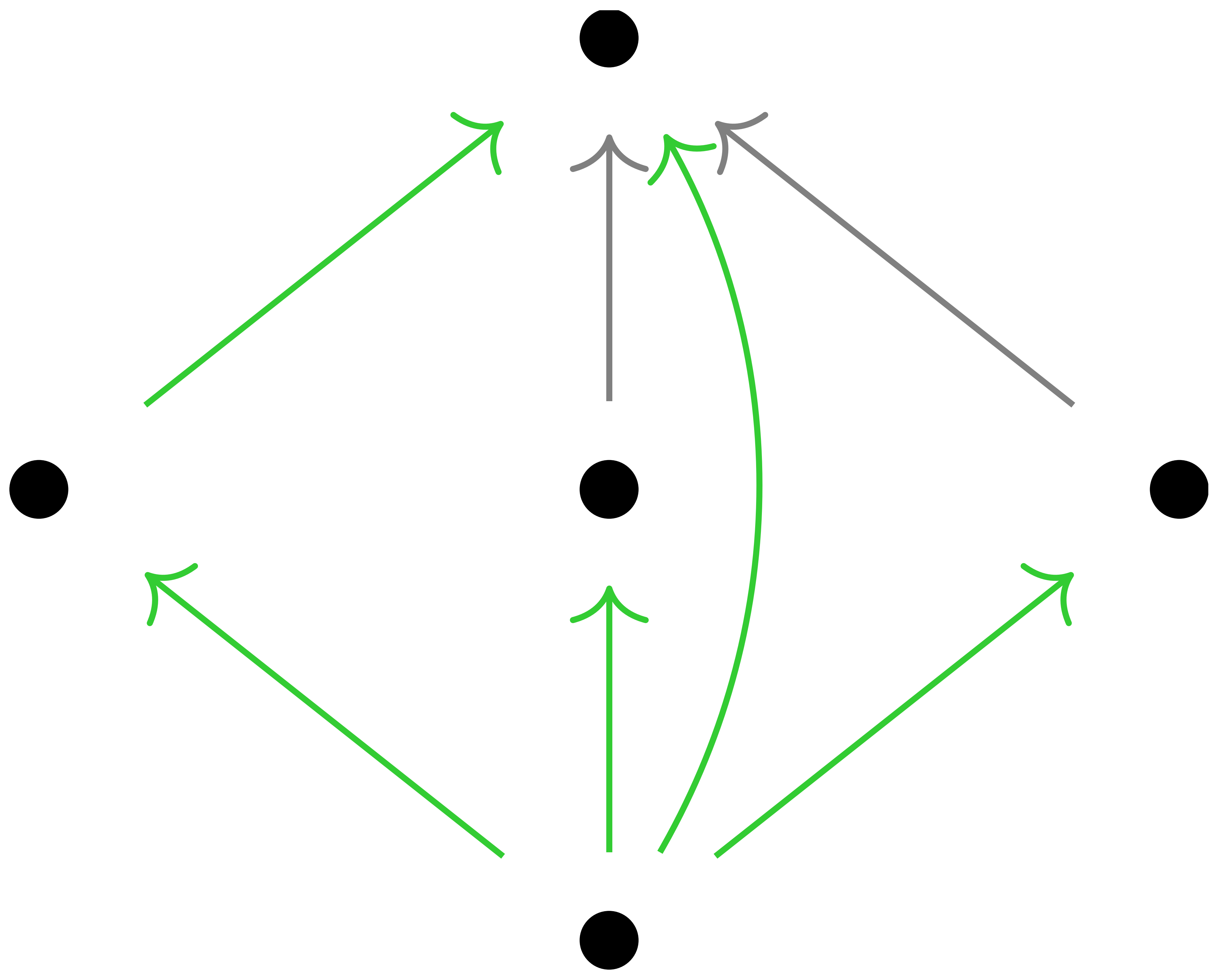
Chapter 2

The Lattice of Transfer Systems

Let L be a finite lattice equipped with an action of a finite group G.

A *transfer system* on L is a wide subposet of L which is G-invariant and closed under pullback.



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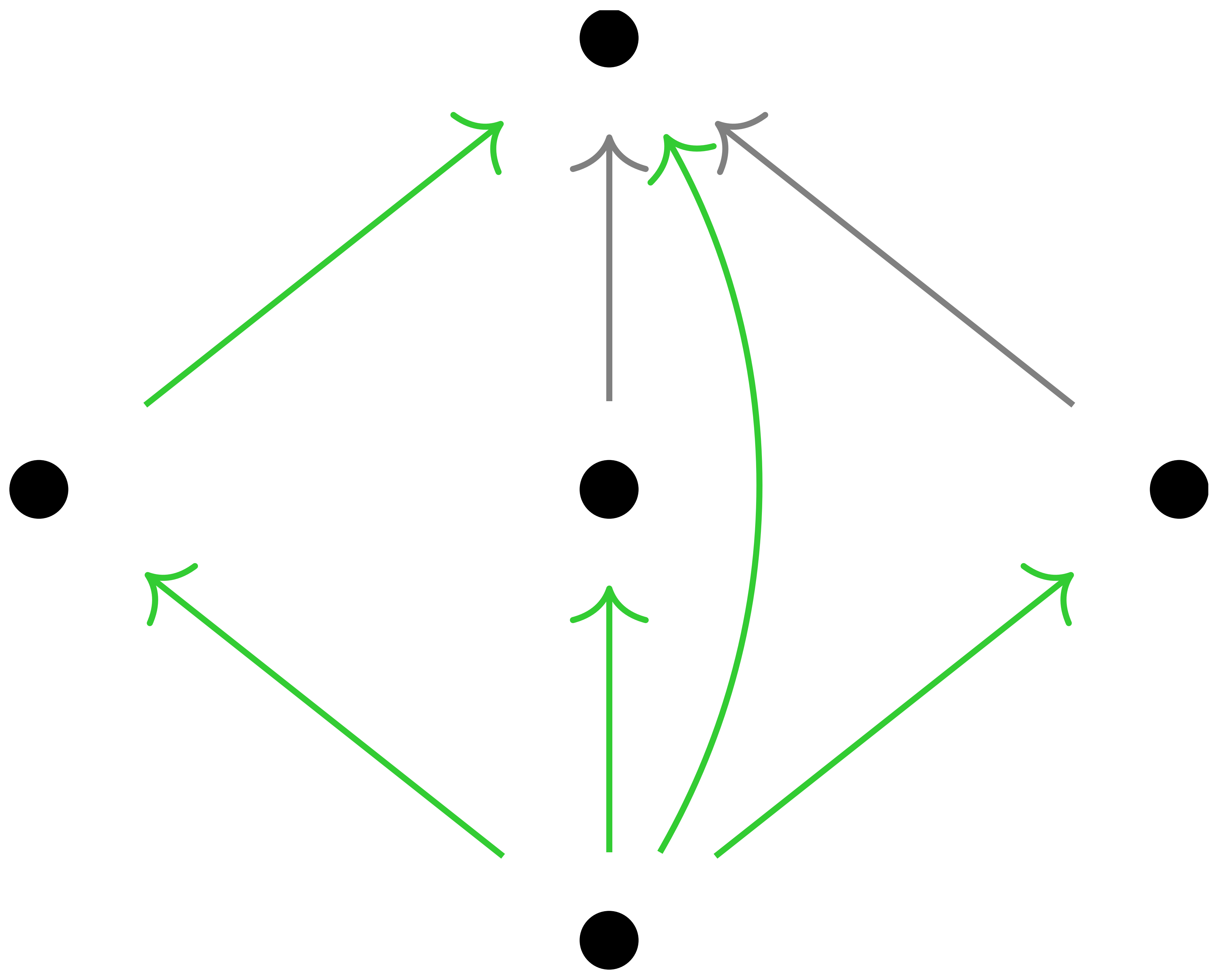
The set of transfer systems on L forms a finite poset $\mathrm{Tr}(L)$, which also turns out to be a lattice.



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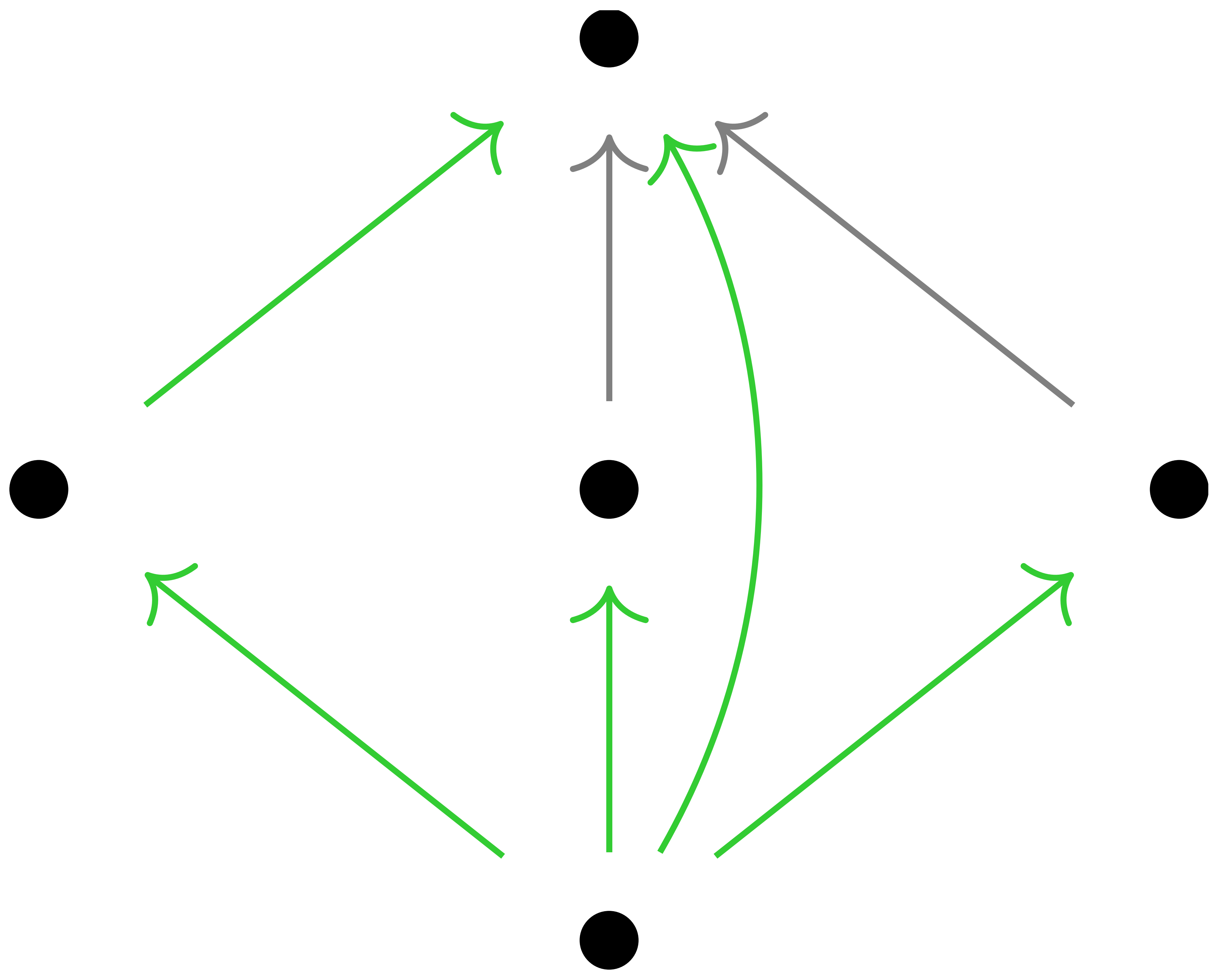
A central problem in *homotopical combinatorics* is to compute the lattice $\mathsf{Tr}(\mathsf{Sub}(G))$ for various finite groups G.



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Example:

$\mathrm{Tr}(\mathrm{Sub}(C_{p^n}))$ is the Tamari lattice T_{n+1}

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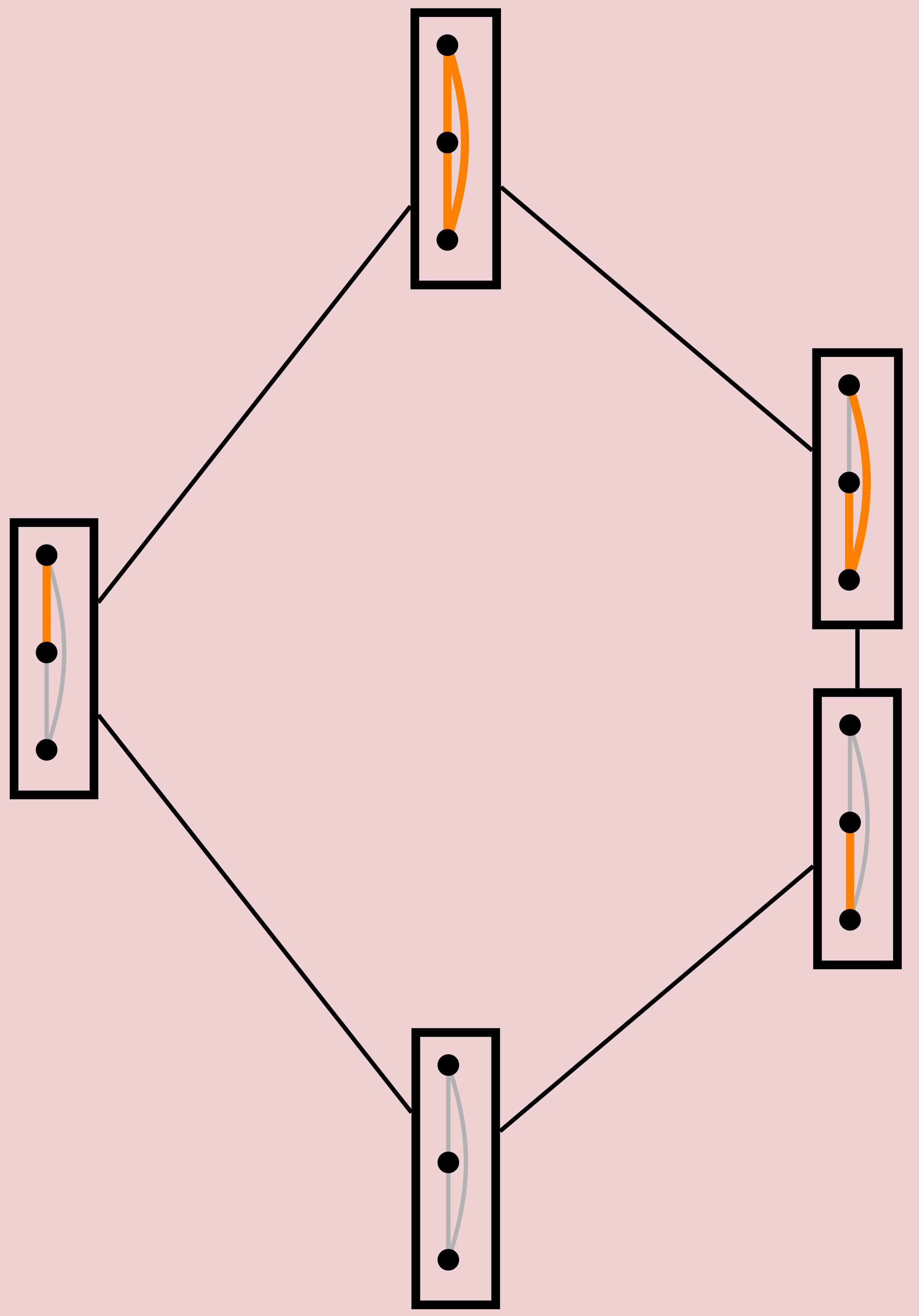

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The Plan To enumerate the elements of $\mathsf{Tr}(L)$, we can:

1. Determine $J(\mathsf{Tr}(L))$ and $M(\mathsf{Tr}(L))$
2. Determine the relation $\{(j, m) \in J(\mathsf{Tr}(L)) \times M(\mathsf{Tr}(L)) : j \leq m\}$
3. Plug this matrix into some FCA software
4. Profit!

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The Lattice of Transfer Systems

Theorem

(Balchin-S.)

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Theorem (Balchin-S.) Let L be a finite lattice with G-action.

$J(\mathsf{Tr}(L))$ and $M(\mathsf{Tr}(L))$ are both in (canonical) bijection with the set of orbits of non-identity arrows in L.

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$$J(\mathsf{Tr}(L)) = \{\lfloor x \to y \rfloor : x < y\} \qquad M(\mathsf{Tr}(L)) = \{\lceil x \to y \rceil^{\perp} : x < y\}$$

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$$\lfloor x \to y \rfloor \leq \lceil x' \to y' \rceil^{\perp} \iff \forall g \in G(g \cdot x \ngeq x' \text{ or } g \cdot y \ngeq y' \text{ or } g \cdot x \geq y')$$

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2. Determine the relation $\{(j,m) \in J(\mathsf{Tr}(L)) \times M(\mathsf{Tr}(L)) : j \leq m\}$ ✅

3. Plug this matrix into some FCA software

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The Lattice of Transfer Systems

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Sub(S_5) $\longrightarrow$

$\longrightarrow$ 183,598,202 transfer systems

Chapter 2

The Lattice of Transfer Systems

Sub(A_6) $\longrightarrow$

$\longrightarrow$ 37,799,146,070 transfer systems

Epilogue

Performance

Lattice	Code from [Bal25]			Code from [BS25]		
	T_1	T_2	$T_1 + T_2$	T_1	T_2	$T_1 + T_2$
$\mathsf{Tr}(\mathsf{Sub}(C_{2^{10}}))$	3.593s	4.567s	≈8s	1.092s	0.007s	≈1s
$\mathsf{Tr}(\mathsf{Sub}(C_2^3))$	2.653s	638.216s	≈11min	0.801s	0.267s	≈1s
$\mathsf{Tr}(\mathsf{Sub}(C_{2^3.3^3}))$	4.005s	1132.035s	≈19min	0.872s	1.138s	≈2s
$\mathsf{Tr}(\mathsf{Sub}(S_5))$	1086.861s	∞	∞	1.275s	11.478s	≈13s
$\mathsf{Tr}(\mathsf{Sub}(A_6))$	>24 hours	∞	∞	2.471s	8063.045s	≈2.2 hours