Algebraic K-theory + even filtration

Conj: (Burklund-Krause)
Let k be a field. Then

$$\mathrm{fil}_{\mathrm{mot}} K(k)^\wedge_\ell \simeq \mathrm{fil}_{\mathrm{ev}} K(k)^\wedge_\ell \,;$$

that is, the motivic and even filtrations on ℓ-adic algebraic K-theory of k coincide.

Today:

1) algebraic K-theory (of fields)
2) the motivic filtration (Voevodsky, ...)
3) the even filtration (Hahn-Raksit-Wilson)

Thm (R.)

The Burklund-Krause conjecture is true for global and local fields.

number fields + function fields of curves over finite fields

completions of global fields

4) What's special about local and global fields? + work of Positselski.

Algebraic K-theory

$X/_k$ smooth variety over a field k

$$\Omega^\infty K(X) \simeq \left(\mathrm{Vect}(X)^{\simeq}\right)^{\text{group complete under } \oplus}$$

K-theory spectrum

groupoid of vector bundles and iso.

connective spectra $\underset{\Omega^\infty}{\simeq}$ ∞-groupoid with commutative multiplication + inverses.

Algebraic K-theory of fields

- $\pi_0 K(k) \simeq \mathbb{Z}$ every vector space is free
- $\pi_1 K(k) \simeq k^\times$ determinant is the universal additive invariant
- $\pi_2 K(k) \simeq k^\times \otimes_{\mathbb{Z}} k^\times / a \otimes (a-1)$ for all $a \neq 0, 1$.

theorem of Matsumoto

- $\pi_n K(k)$ for $n > 2$?

complicated, related to special values of L-functions (Beilinson)

Ex (Quillen, finite fields)

$$K_{2i}(\mathbb{F}_q) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & \text{otherwise} \end{cases}$$

$$K_{2i-1}(\mathbb{F}_q) \cong \mathbb{Z}/(q^i-1)\mathbb{Z} .$$

Ex (Borel, the rationals)

$$\mathbb{Q} \underset{\mathbb{Z}}{\otimes} K_n(\mathbb{Q}) \cong \begin{cases} \mathbb{Q} & n=0 \\ \bigoplus_{p \text{ prime}} \mathbb{Q} & n=1 \\ \mathbb{Q} & n=1 \bmod 4,\ n\geq 5 \\ 0 & \text{otherwise} \end{cases}$$

X/k variety

$K(X)$

rational K-theory

$\mathbb{Q}\otimes K(X)$

- satisfies Galois descent
- somewhat for Ran topology

ℓ-adic K-theory

$K(X)^\wedge_\ell$, $\ell \neq \mathrm{char}\, k$

- close to étale cohomology
- „topological"

p-adic K-theory

$K(X)^\wedge_p$ $p = \mathrm{char}\, k$

- can be described in terms of differential forms /
- trace methods

$\underline{\text{Ex:}}$ $K_1(\mathbb{C}) \simeq \mathbb{C}^\times \simeq \mathbb{Q}/\mathbb{Z} \oplus V$

roots of unity

uncountable rational vector space

rationalize

V extremely large!

ℓ-adic completion

$\mathbb{Z}_\ell$ (in degree 1)

$\underline{\text{Thm:}}$ (Suslin)

$$K(k^{sep})^\wedge_\ell \simeq ku^\wedge_\ell$$

chromatic redshift

ℓ-adic K-theory of separably closed fields doesn't depend on the field

Thm: (Thomason)

$$X \longmapsto K(X)^\wedge_\ell[\beta^{-1}] \simeq L_{K(1)} K(X)$$

satisfies étale descent.

Motivic cohomology

M smooth mfd: Atiyah-Hirzebruch sseq

$$H^p(X, \mathbb{Z}(q)) \Rightarrow KU^{2q-p}(X)$$

singular cohomology

Here

$\mathbb{Z}(q) \simeq \pi_{2q} KU \simeq \mathbb{Z}$.

comes from the filtration $(\tau_{\geq 2*} KU)^M$ on KU^M

X/k smooth variety: the motivic spectral sequence

$$H^p_{mot}(X, \mathbb{Z}(q)) \Rightarrow K_{2q-p}(X)$$

Voevodsky, Bloch, Lichtenbaum, Friedlander, Suslin, Levine

motivic cohomology (built out of algebraic cycles)

comes from the motivic filtration

$\mathrm{fil}_{mot} K(X)$

The norm residue isomorphism

Thm. (Voevodsky, Rost)

$$H^p_{mot}(X, \mathbb{Z}/c(q)) \simeq \begin{cases} H^p_{et}(X, \mathbb{Z}/c(q)) & p \leq q \\ 0 & \text{otherwise} \end{cases}$$

Ex ($X = \mathrm{Spec}(k)$)

$H^p_{et}(k, \mathbb{Z}/c) \simeq H^p(G, \mathbb{Z}/c)$ group cohomology

$G = \mathrm{Gal}(k^{sep}/k)$

absolute Galois group

Intuitively, this is saying that c-adically K-theory depends only on the Galois group (Bachmann - Burklund - Xu).

Cor:

$$\bigoplus_p H^p(G, \mathbb{Z}/c(p)) \simeq \bigoplus_p H^p_{mot}(G, \mathbb{Z}/c(p)) \simeq \bigoplus_p K^M_p(k)/c$$

is a quadratic algebra.

The even filtration

$A \in \mathrm{CAlg}(\mathrm{Sp})$ $\quad x \in \pi_n A,\ y \in \pi_m A \quad xy = \underline{(-1)}^{m \cdot n} y x$

the sign can be ignored when m, n are even

Def: (Hahn-Raksit-Wilson)

$$\mathrm{fil}^{*}_{ev} A = \lim_{\substack{\longleftarrow \\ A \to B \\ \pi_* B \text{ even}}} \tau_{\geq 2*} B$$

Can be calculated using a combination of:

- If $\pi_* A$ even, then $\mathrm{fil}^{*}_{ev} A \simeq \tau_{\geq 2*} A$
- $\mathrm{fil}^{*}_{ev}(-)$ satisfies a form of faithfully flat descent

Examples

due to Hahn-Raksit-Wilson

$A \in \mathrm{CAlg}(\mathrm{Sp})$	$\mathrm{fil}^*_{ev}(A)$	E_2-term
$\mathbb{S}$ the sphere	Adams-Novikov filtration	coh. of the moduli stack of formal groups
$\mathrm{THH}(R/k)$	Hochschild-Kostant-Rosenberg	diff. forms
$\mathrm{TP}(R)^{\wedge}_p$	Bhatt-Morrow-Scholze	prismatic cohomology

HRW show that $\mathrm{fil}^*_{ev}\mathrm{TP}(-)$ gives a reasonable notion of prismatic cohomology of ring spectra.

What about algebraic K-theory?

Thm. (Raksit)

$$\mathrm{fil}_{\mathrm{mot}} K(\mathbb{F}_q)^\wedge_\ell \simeq \mathrm{fil}_{\mathrm{ev}} K(\mathbb{F}_q)^\wedge_\ell$$

Thm. (Burklund–Krause)

If A is connective, then

1) $\mathrm{gr}^q_{\mathrm{ev}} A$ is q-connective
2) $\bigoplus_{q \in \mathbb{Z}} \pi_q \mathrm{gr}^q_{\mathrm{ev}}(A)$ is a quadratic algebra.

Cor: The even and motivic filtrations on $K(k)$ satisfy the same connectivity bound and agree in the lowest degree.

Conj: (Burklund–Krause)

Let k be a field. Then

$$\mathrm{Fil}^{*}_{\mathrm{ev}} K(k)^{\wedge}_{C} \simeq \mathrm{Fil}^{*}_{\mathrm{mot}} K(k)^{\wedge}_{C}.$$

Thm. (P.)

The B-K conjecture is true for

- global fields
- local fields
- fields of $\mathrm{cd} \leq 1$ (e.g. $\mathbb{C}(t)$)

Cor: For above fields, there exists a mod C abelian category of mixed motives.

A word on the proof

Conj. (Positselski's Koszulity conjecture)
Let k be a field containing the ℓ-th root of unity.
Then $K^M_*(k)/\ell$ is a Koszul algebra.

$\cong$

$H^*(G, \mathbb{Z}/\ell)$

- generated in deg 1
- relations in deg 2
- relations between relations in deg 3
- ...

The key observation is that when k satisfies the Koszulity conjecture then $K(k)^\wedge_\ell \to K(k^{sep})^\wedge_\ell$ is faithfully even flat.

Thm: (Positselski)
The Koszulity conjecture is true for local and global fields.