Multiplicative Negative K-Theory and Quantum Cellular Automata

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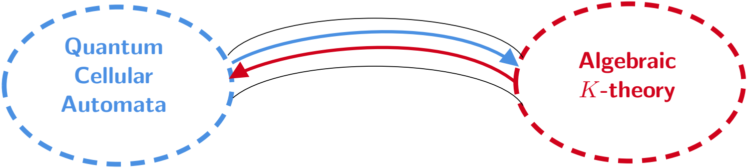
This talk is about two seemingly disjoint subjects:

1. Quantum Cellular Automata (QCA) from quantum computing.
2. Algebraic K-theory from homotopy theory.

Quantum Cellular Automata

Algebraic K-theory

... and a bridge between the two subjects.



Outline

The Heisenberg Picture for Quantum Mechanics

Let V be a:

- finite dimensional complex Hilbert space (i.e., $\mathbb{C}^n$)

The **Heisenberg picture** for quantum mechanics treats a quantum system as a non-commutative algebra of observables

$$\mathcal{A} := \mathrm{End}(V).$$

$\mathcal{A}$ has an adjoint operation $*$ coming from the Hermitian inner product.

Symmetries of quantum systems are then modeled as $*$-preserving automorphisms of $\mathcal{A}$, which is precisely

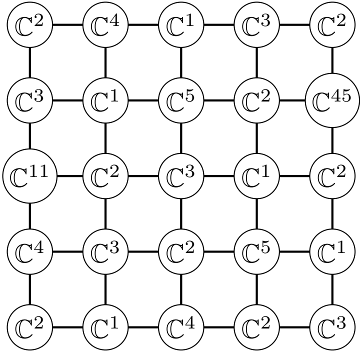
the projective unitary group $PU(n)$.

Many-Particle Quantum Systems

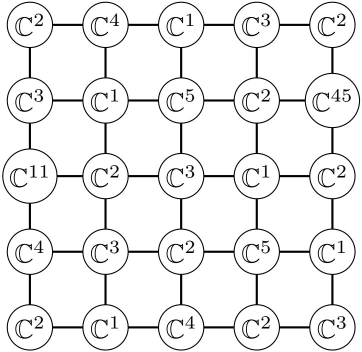
The discussions prior can be viewed as single-particle quantum mechanics:

$$V = \mathbb{C}^n$$

Often times, one would like to model quantum systems on multiple (and even infinitely many) particles.



What is Quantum Cellular Automata



In the many-body case over a lattice X,

1. The algebra of observables $\mathcal{A}(X, q)$ becomes a colimit of matrix algebras.
2. Quantum cellular automata (QCA) is the physically appropriate automorphism group of $\mathcal{A}(X, q)$.

What is a QCA, More Formally.

Throughout this talk, we will let:

1. R be a commutative ring with unity.
2. X be a nice metric space[1], such as $\mathbb{Z}^n$ with the L^∞-metric or lattices in Lie groups.

The placement of vector spaces (or free R-modules) on X is given by the following.

Definition:

A quantum spin system is a set function

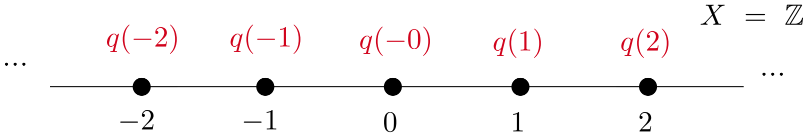
$$q : X \to \mathbb{Z}_{\geq 1} = \{1, 2, 3, \dots\}$$

such that for each $y \in X$ and $r \geq 0$,

$$B_r(y) \cap \{x \in X : q(x) > 1\} \text{ is finite.}$$

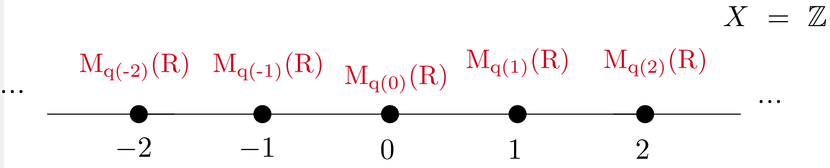
[1] We assume X contains countable, coarsely dense subspace that is uniformly discrete, locally finite metric space of bounded geometry.

Quantum Spin System



Algebra of Observables

Since we only care about the endomorphisms, morally we might as well think of a quantum spin system as the picture:



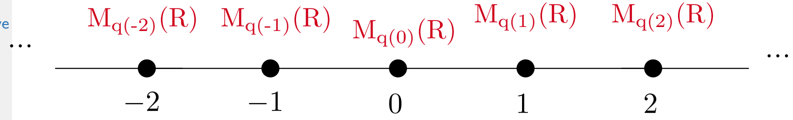
Here $M_n(R)$ is the noncommutative R-algebra of $n \times n$ matrices over R.

Informal:

The algebra of observables $\mathcal{A}(X, q)$ is the "infinite tensor product" of $M_{q(x)}(R), x \in X$.

Algebra of Observables

$X = \mathbb{Z}$



More precisely,

$$\mathcal{A}(X,q) := \operatorname*{colim}_{\text{finite subsets } S \subseteq X} \underbrace{\bigotimes_{s\in S} M_{q(s)}(R)}_{=:\mathcal{A}(S,q)},$$ [2]

where for $S \subseteq T$ finite subsets,

$$\mathcal{A}(S,q) \to \mathcal{A}(T,q)$$

is given by $- \otimes \mathrm{id}_{\mathcal{A}(T-S,q)}$, tensoring with the identity operator on $T - S$.

[2] Algebras like these are examples of what are called supernatural algebra [Bar-On et al., 2023].

Quantum Cellular Automata

A quantum cellular automation α is an R-algebra automorphism

$$\alpha : \mathcal{A}(X, q) \to \mathcal{A}(X, q)$$

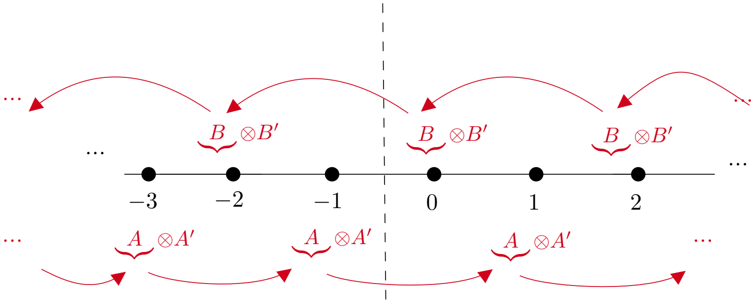
of finite spread, meaning that there exists some $\ell > 0$ such that for all $\gamma \in M_{q(x)}(R)$, $\alpha(\gamma) \in \mathcal{A}(D_\ell(x), q)$.

In other words, a QCA is an "automorphism that moves components away in a distance that is uniformly bounded".

Definition:

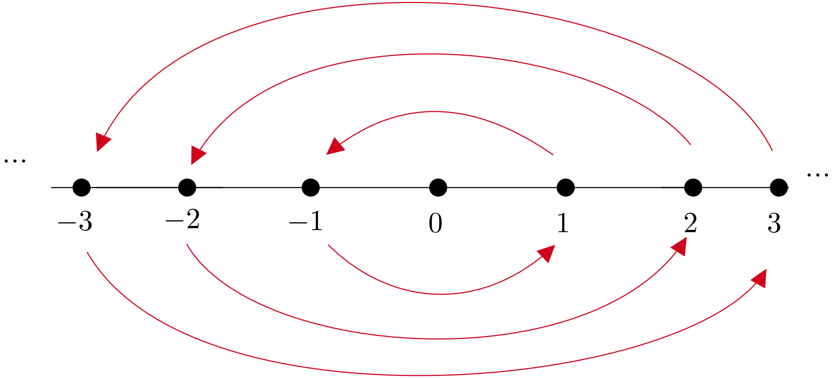
The group of QCAs over $\mathcal{A}(X, q)$ is denoted $\mathcal{Q}(X, q)$.

Example of QCA



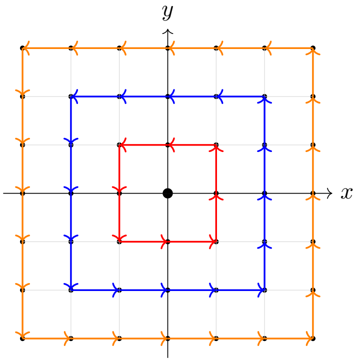
A QCA α for a pair (A, B) of Azumaya R-algebras, B' and A' are tensor complements such that $A \otimes A' \cong M_n(R)$ and $B \otimes B' \cong M_m(R)$.

Not An Example of QCA



Here $q : X \to \mathbb{Z}_{>0}$ is constant, the antipodal map swaps the tensor factors.

Example of 2D QCA



Annular shifts on the plane. This has spread 1.

The Total QCA Group

We can stabilize the QCA groups from earlier.

1. For quantum spin systems $q, r : X \to \mathbb{Z}_{\geq 1}$, we say $q \leq r$ if
$$q(x) \text{ divides } r(x), \quad \forall x \in X.$$
2. For $q \leq r$, we can define a map
$$\mathcal{Q}(X, q) \to \mathcal{Q}(X, r)$$
by pointwise tensor product with the identity matrix of dimension $r(x)/q(x)$.
3. The total QCA group is
$$\mathcal{Q}(X) := \operatorname{colim}_q \mathcal{Q}(X, q).$$

Definition:

The circuit subgroup of $\mathcal{Q}(X)$ is the commutator subgroup $\mathcal{C}(X) = [\mathcal{Q}(X), \mathcal{Q}(X)]$[3].

A central question in QCA is:

Question:

How can we classify QCAs up to circuits?

[3] The circuit subgroup has physical meaning and was defined differently, which is omitted for the sake of time. In [Ji and Yang, 2026b], we showed they are equivalent in most cases of interest.

Question:

How can we classify QCAs up to circuits?

The groups $\mathcal{Q}(X)^{ab} := \mathcal{Q}(X)/\mathcal{C}(X)$ are infamously difficult to compute! In the (unitary) case over $\mathbb{C}$:

1. $\mathcal{Q}(\mathbb{Z}^1)^{ab} = \mathbb{Q}_{>0}$ GNVW index [Gross et al., 2012].
2. $\mathcal{Q}(\mathbb{Z}^2)^{ab} = 0$ by a theorem of Michael Freedman and Matthew Hasting [Freedman and Hastings, 2020].
3. Jeongwan Haah recently obtained a proof that $\mathcal{Q}(\mathbb{Z}^3)^{ab}$ contains a copy of $\mathbb{Z}/2\mathbb{Z}$.
4. The higher groups are difficult in general.

It has been conjectured [Haah, 2021] that $\mathcal{Q}(\mathbb{Z}^3)^{ab}$ is related/equal to the Witt group of braided fusion $\mathbb{C}$-linear categories.

Why Homotopy Theory?

Influenced by Kitaev's conjecture of Ω-spectrum for invertible states, there has been a folklore conjecture in the field that the classification of QCA should be governed by an Ω-spectrum.

This has also been mentioned explicitly in [Long and Else, 2025, Tu et al., 2025].

$$\begin{array}{ccc} & & \textbf{QCA Groups} \\ & & \rightsquigarrow \\ \textbf{Homotopy Theory} & \rightsquigarrow & \textbf{QCA Spaces and Spectra} \\ & & \rightsquigarrow \\ & & \textbf{Computations} \end{array}$$

Homotopical Classifications for QCAs

The QCA Conjecture

For each spatial dimension d, there is a space-level construction $\mathbf{Q}(\mathbb{Z}^d)$ of the QCA classification group with a canonical isomorphism

$$\pi_0\mathbf{Q}(\mathbb{Z}^d) \cong \mathcal{Q}(\mathbb{Z}^d)/\mathcal{C}(\mathbb{Z}^d).$$

Moreover, these spaces assemble naturally into an Ω-spectrum $\mathbb{QCA} = \{\mathbf{Q}(\mathbb{Z}^d)\}_{d\geq 0}$, satisfying certain expected properties.

The QCA Conjecture

Theorem ([Ji and Yang, 2026b])

The QCA conjecture is true over any ring R. Furthermore, for any nice metric space X, there is an analogous Ω-spectrum of the form

$$\mathbb{QCA}(X): \ \mathbf{Q}(X), \mathbf{Q}(X \times \mathbb{Z}), \mathbf{Q}(X \times \mathbb{Z}^2), \dots$$

1. The QCA conjecture is the special case when $X = *$ is a single point.
2. When $R = \mathbb{C}$, this can be adapted to the unitary case at the beginning to give an unitary QCA spectrum.

Our strategy uses a subject called algebraic K-theory.

Outline

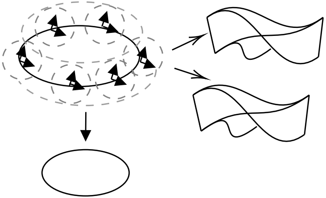
Question:

What is Algebraic K-theory?

On a high level, K-theory is a general methodology / abstraction of the process of:

1. Breaking an object apart.
2. Assembling the pieces together.
3. Studying structural properties from the additive data.

Conceptual Illustration

For a ...	*The objects are ...*	*We decompose by ...*	
Field k	Finite Dimensional k-Vector-Spaces	Direct Sums	$\mathbb{R}^4 \to \mathbb{R}^1$, $\mathbb{R}^1$, $\mathbb{R}^2$
Ring R	Finitely Generated Project R-Modules	Direct Sums	$\mathbb{Z}/6\mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$
Topological Space X	Finite Rank Vector Bundles on X	Whitney Sums	
Symmetric Monoidal Category C	Objects in C	The tensor product $\otimes$	$\{1,3,4,17\} \to \{1\}$, $\{3,17\}$, $\{4\}$

What is Algebraic K-Theory

Algebraic K-theory is a functor

$$\text{Symmetric Monoidal Categories} \xrightarrow{K(-)} \text{Spc} .$$

For a symmetric monoidal category $\mathcal{S}$, its i-th K-theory is:

$$K_i(\mathcal{S}) := \pi_i K(\mathcal{S}).$$

Example:

For a ring R, let $\mathrm{Proj}(R)^{f.g.}$ denote the symmetric monoidal groupoid of finitely generated projective R-modules, with direct sum as the operation.

The K-theory of R is $K(R) := K(\mathrm{Proj}(R)^{f.g.})$.

Negative K-Theory

The definition above accounts for the higher K-theory of R.

Bass considered instead the lower K-groups that he called negative K-theory.

1. These are groups $K_{-i}(R)$ for $i \geq 0$.
2. They are useful invariants in detecting whether or not a ring is smooth, among other applications.

Definition:

Classicially, they are defined as follows:

1. $K_{-1}(R) := \mathrm{coker}(K_0(R[t]) \oplus K_0(R[t^{-1}]) \to K_0(R[t^{\pm 1}]))$.
2. Since the construction is functorial, one iteratively defines

$$K_{-(n+1)}(R) := \mathrm{coker}(K_{-n}(R[t]) \oplus K_{-n}(R[t^{-1}]) \to K_{-n}(R[t^{\pm 1}]))$$

The Pedersen-Weibel Construction

Question:

Can the groups $K_{-i}(R)$ be realized as the negative homotopy groups of a non-connective delooping of the space $K(R)$?

[Pedersen and Weibel, 1985] gave a concrete way to construct this, which will look quite familiar.

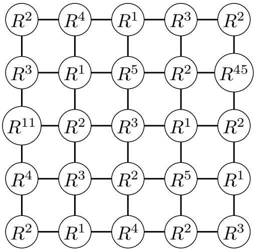
The Pedersen-Weibel Construction

For each $i \geq 0$, the Pedersen-Weibel bounded category $\mathcal{C}_i(R)$ is a symmetric monoidal category whose:

- Objects are $\bigoplus_{x \in \mathbb{Z}^i} R^{q(x)}$ for $q : \mathbb{Z}^n \to \mathbb{Z}_{\geq 0}$.
- Morphisms are R-linear isomorphisms

$$\phi_{x,y} : R^{q(x)} \to R^{q(y)}$$

  such that $\phi_{x,y} \equiv 0$ if $d(x, y) > \ell$.
- Symmetric Monoidal under pointwise direct sum.



The Pedersen-Weibel Delooping

Theorem ([Pedersen, 1984])

For $i \geq 0$, $K_{-i}(R) = K_1(\mathcal{C}_{i+1}(R))$.

The construction of PW bounded category in fact works for any additive category.

Theorem ([Pedersen and Weibel, 1985], Stated Informally.)

After a suitable idempotent completion procedure, the spaces $K(\mathcal{C}_i(R))$ for $i \geq 0$ form a non-connective delooping of $K(R)$, denoted $\mathbb{K}(R)$.

Clearly Pedersen and Weibel's construction works for any nice metric space X, this was studied in a follow-up paper [Pedersen and Weibel, 1989].

This perspective has been developed into what is called controlled K-theory [Carlsson and Goldfarb, 2011].

Outline

For the input (X, R), we define a QCA category $\mathbf{C}(X; R)$ whose:

- Objects are $\mathcal{A}(X, q)$.
- Morphisms of R-algebra isomorphisms of finite spread. [4].
- Symmetric monoidal under pointwise tensor product $\otimes$.

Example:

- When $X = *$, $\mathbf{C}(*; R)$ is the symmetric monoidal category of matrix rings $M_n(R)$, under tensor product.

$$K_0(\mathbf{C}(*; R)) \cong \mathbb{Q}_{>0}.$$

[4] Importantly, $\mathcal{C}(X; R)$ is not usually skeletal

Structural Properties of the QCA Category

Theorem ([Ji and Yang, 2026b])

1. When $\operatorname{Spec} R$ is connected, $K_0(\mathbf{C}(X;R)) \cong \mathrm{CH}_0(X;\bigoplus_{\mathbb{N}}\mathbb{Z})$, the 0-th coarse homology group of X in coefficient $\bigoplus_{\mathbb{N}}\mathbb{Z}$.
2. For $i \geq 1$, $K_0(\mathbf{C}(\mathbb{Z}^i;R)) = 0$.
3. $K_1(\mathbf{C}(X;R)) = \mathcal{Q}(X)/\mathcal{C}(X)$.
4. $K_1(\mathbf{C}(\mathbb{Z}^1);R) = K_0(\mathrm{Az}_R,\otimes)$, where Az_R is the category of Azumaya R-algebras.

The QCA classification group being at π_1 motivated[5] us to take a loop space.

Definition

The QCA space of $(X;R)$ is

$$\mathbf{Q}(X;R) \coloneqq \Omega(K(\mathbf{C}(X;R))_1.$$

From here, our main theorem was:

Theorem ([Ji and Yang, 2026b])

There is a canonical delooping

$$\mathbf{Q}(X;R) \cong \Omega\mathbf{Q}(X \times \mathbb{Z};R).$$

[5] This was suggested to us by Tomer Schlank.

Conceptual Challenges

Theorem ([Ji and Yang, 2026b])

There is a canonical delooping

$$\mathbf{Q}(X;R) \cong \Omega\mathbf{Q}(X \times \mathbb{Z};R).$$

There are certain challenges in proving this that Pedersen-Weibel delooping did not encounter:

1. The category $\mathbf{C}(X;R)$ is Not Additive.
2. Idempotent completion does not really make sense for $\mathbf{C}(X;R)$. For direct sums, there is an obvious projection
$$\pi_A : A \oplus B \to A.$$
The analog of such projection does not really make sense for
$$\pi_A : A \otimes B \to A.$$
3. We cannot split an element into its individual parts at each point $x \in X$.

Proof Idea

Theorem ([Ji and Yang, 2026b])

There is a canonical delooping

$$\mathbf{Q}(X; R) \cong \Omega\mathbf{Q}(X \times \mathbb{Z}; R).$$

Proof: The proof required a lot of soul-searching in algebraic topology and noncommutative algebra.

Consequences

1. In the proof of the main theorem, we showed something stronger that

$$\Omega K(\mathbf{C}(X \times \mathbb{Z}; R)) = K(\text{Invertible } R\text{-Subalgebras of } X)$$

which generalized a theorem of [Haah, 2023] when we take π_0 on both sides.

2. Using this, we could actually identify

$$\mathbf{Q}(\mathbb{Z}^1; R) = K(\mathrm{Az}_R, \otimes).$$

Thus, the QCA spectrum $\mathbb{QCA}(*; R)$ can also be seen as a non-connective delooping of $K(\mathrm{Az}_R, \otimes)$.

3. For each X, $\mathcal{Q}(X)/\mathcal{C}(X) \cong H_2(\mathcal{C}(X \times \mathbb{Z}^1); \mathbb{Z})$.

Table of Comparisons

Algebraic K-theory	Quantum Cellular Automata
Free R-Modules R^n	Matrix R-Algebras $M_n(R)$
Projective R-Modules	Azumaya R-Algebras
Direct Sum $\oplus$	Tensor Product $\otimes$
Reduced $\widetilde{K}_0(R)$	Brauer Group $\mathrm{Br}(R)$
Negative K-theory	QCA Groups
The spectrum $\mathbb{K}(R)$	The spectrum $\mathbb{QCA}(*; R)$
Controlled K-theory	$\mathbb{QCA}(X; R)$

Algebraic K-theory	Quantum Cellular Automata
Free R-Modules R^n	Matrix R-Algebras $M_n(R)$
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Controlled K-theory	$\mathbb{QCA}(X; R)$

Thematic Questions:

What can we say about QCA groups using techniques from K-theorists?

What can we say about multiplicative K-theory using techniques from the quantum physicists?

More Information

If you are interested to see more, check out our papers:

Initial paper resolving the QCA Conjecture:

Ji, M. and Yang, B. (2026b). Quantum Cellular Automata: The Group, the Space, and the Spectrum (https://arxiv.org/abs/2602.16572)

Follow-up paper on applications to linearization and onsiteability:

Ji, M. and Yang, B. (2026a). K-Theoretic Obstructions to Linearizing QCA Representations (https://arxiv.org/abs/2606.19657)

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