MINI-COURSE: CONFIGURATION SPACES

These are lecture notes for a mini-course given by Ben Knudsen at the *Algebraic structures in topology II* conference that took place June 5-14, 2024 in San Juan, Puerto Rico. They were typed by Sarah Anderson and Fabio Capovilla-Searle. Any typos or mistakes are the fault of the scribers (**not** the speaker).

1. Lecture 1

Definition 1.1. *Let X be a space. The ordered Configuration space of k many points in X is*

$$F_k(X) = \{(x_1, \ldots, x_k) \in X^k \mid x_i \neq x_j \text{ if } i \neq j\}.$$

The unordered Configuration space of k many points in X is

$$B_k(X) = F_k(X)/\Sigma_k,$$

where Σ_k is the group of permutations on k elements.

Axiom 1.2. *Configuration Spaces are intrinsically interesting.*

Question. How do we study them?

There are two ways: one at a time or inductively. We start with studying them one at a time.

Example. Let $X = \mathbb{R}^n$ and $k = 2$, that is, 2 labeled points in $\mathbb{R}^n$ as in figure 1.

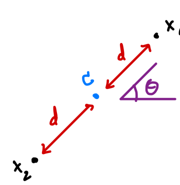

Figure 1. $F_2(\mathbb{R}^n)$

There are three relevant pieces of information for which we can always find the labeled points: the center of mass c, the distance d of each point from the center of mass, and the direction θ of the two labeled points with respect c. This results in

$$F_2(\mathbb{R}^n) \cong \mathbb{R}^n \times \mathbb{R}_{>0} \times S^{n-1}$$

Since $\mathbb{R}^n$ and $\mathbb{R}_{>0}$ are contractible, then

$$F_2(\mathbb{R}^n) \xrightarrow[\sim]{\gamma} S^{n-1}$$

$$(x_1, x_2) \mapsto \frac{x_2 - x_1}{|x_2 - x_1|}$$

where γ is the Gauss map. In addition,

$$B_2(R) \simeq \mathbb{R}P^{n-1}$$

Example. Let $X = \mathbb{R}$ and k an arbitrary integer as in figure 2.

FIGURE 2. $F_k(\mathbb{R})$

Then

$$\begin{aligned} B_k(\mathbb{R}) &= \{(t_1, \dots, t_k) \in [0,1] \mid 0 < t_1 < \dots < t_k < 1\}, \\ &\cong \operatorname{int}(\Delta^k), \\ &\simeq * \end{aligned}$$

where Δ^k is a k-simplex. In addition,

$$F_k(\mathbb{R}) \simeq \Sigma_k.$$

Example. Let $X = S^n$ and $k = 2$ as in figure 3.

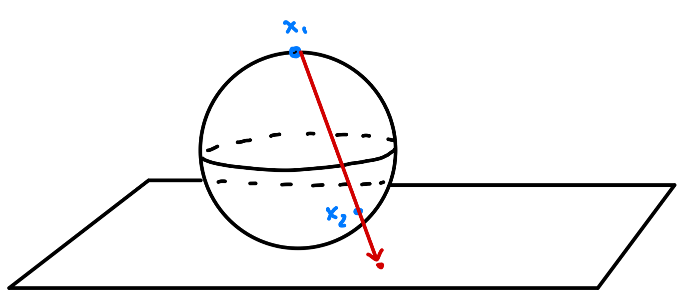

FIGURE 3. $F_2(S^n)$

Stereo-graphic projection gives

$$F_2(S^n) \cong TS^n$$

Example. Let $X = S^1$ and k an arbitrary integer as in figure 4.

Stereo-graphic projection gives

$$F_k(S^1) \simeq \coprod_{(k-1)!} S^1$$

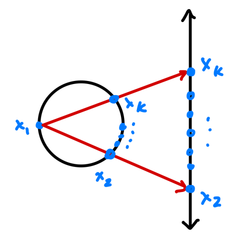

Figure 4. $F_k(S^1)$

The other way of studying Configuration spaces is by induction.

Theorem 1.3. *(Fadell-Neuwirth 1962) Let M be a connected manifold of dimension greater than 1. The restricted coordinate projection*

$$F_k(M) \to F_{k-1}(M)$$
$$(x_1, \dots, x_k) \mapsto (x_1, \dots, x_{k-1})$$

is a fiber bundle with fiber $M \setminus \{x_1, \dots, x_{k-1}\}$.

Corollary 1.4. *If Σ is an aspherical surface, then $F_k(\Sigma)$ and $B_k(\Sigma)$ are also aspherical.*

The fundamental group of $F_k(\Sigma)$ is the pure surface braid groups and the fundamental group of $B_k(\Sigma)$ is the surface braid groups.

Example. Thanks to Artin,

$$\pi_1(B_k(\mathbb{R}^2)) \cong \langle \sigma_1, \dots, \sigma_{k-1} \mid \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}, \sigma_i\sigma_j = \sigma_j\sigma_i \text{ if } |i-j| > 1\rangle$$

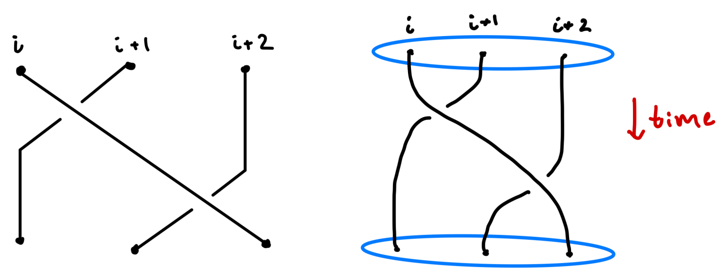

Figure 5. An example of a pure surface braid group.

We can also compute the group cohomology of certain configuration spaces. For example, when $M = \mathbb{R}^n$ the Serre Spectral Sequence is:

$$H^p(F_{k-1}(\mathbb{R}^n); H^q(\mathbb{R}^n \setminus \{x_1, \dots, x_{k-1}\}) \Rightarrow H^{p-q}(F_k(\mathbb{R}^n))$$

By the Leray-Hirsch Theorem, the spectral sequence collapses provided

$$H^*(F_k(\mathbb{R}^n)) \to H^*(\mathbb{R}^n \setminus \{x_1, \dots, x_{k-1}\})$$

is surjective. The fundamental class of the ith sphere comes from $S^{n-1} \cong \partial B_\epsilon(x_i) \subseteq \mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\}$.

$$F_k(\mathbb{R}^n) \xrightarrow{\pi_{ab}} F_2(\mathbb{R}^n) \xrightarrow{\gamma} S^{n-1}$$

Exercise. The degree of

$$S^{n-1} \cong \partial B_\epsilon(x_i) \subseteq \mathbb{R}^n \setminus \{x_1, \ldots, x_{k-1}\} \subseteq F_k(\mathbb{R}^n) \xrightarrow{\gamma_{jk}} S^{n-1}$$

is δ_{ij} (the Kronecker delta).

Surjectivity follows, so the Betti numbers (and homology) of $F_k(\mathbb{R}^n)$ coincide with those of $\prod_{j=1}^{k-1} \vee_j S^{n-1}$. We also learn that the fundamental class $\alpha_{ab} := \gamma_{ab}^* \in H^{n-1}(F_k(\mathbb{R}^n))$ generate the cohomology ring. The fundamental class α_{ab}: has the following properties:

(1) $\alpha_{ab}^2 = 0$.
(2) $\alpha_{ab} = (-1)^n \alpha_{ba}$.
(3) The Arnold relation: $\alpha_{ab}\alpha_{bc} + \alpha_{bc}\alpha_{ca} + \alpha_{ca}\alpha_{ab} = 0$

The proof of (3): reduce to $k = 3$, $(a, b, c) = (1, 2, 3)$ and note that $\mathrm{rk} H^{2(k-1)} = 2$ and use the action of Σ_3.

Definition 1.5. *Let $\mathcal{A}$ be the quotient of the free graded commutative ring on $\{\alpha_{ab}\}_{1 \le a \ne b \le k}$ by relations (1), (2) and (3).*

Theorem 1.6. *(Arnold-Cohen) The map*

$$\mathcal{A} \to H^*(F_k(\mathbb{R}^n))$$

is an isomorphism.

Proof. Leray-Hirsch Theorem gives a basis $\{\alpha_{a_1 b_1}\alpha_{a_2 b_2 2} \ldots \alpha_{a_m b_m}\}$ such that $a_i < b_i$ and $b_1 \le b_2 \le \cdots \le b_m$. An argument with the Arnold relation shows that this set spans $\mathcal{A}$. □

A dual spanning set in homology is given by "planetary systems", i.e., plannar rooted forests with k leaves, i.e., parenthesized partitions of $\{1, \ldots, k\}$. This can be seen in figure 6.

The dual relation to the Arnold relation is

$$((12)3) + ((23)1) + ((31)2) = 0$$

which is a Jacobi relation.

Corollary 1.7. *For $k \ge 2$,*

$$H_i(B_k(\mathbb{R}^n); \mathbb{Q}) \cong \begin{cases} \mathbb{Q} & i = 0 \text{ or } i = n-1 \text{ odd} \\ 0 & \text{otherwise} \end{cases}$$

Proof. Figure 7 is a proof by picture. □

Or if $B = \coprod B_k$ then

$$H_*(B(\mathbb{R}^n); \mathbb{Q}) \cong \begin{cases} P[x] \otimes \Lambda[y] & \text{if } n \text{ is even} \\ P[x] & \text{if } n \text{ is odd} \end{cases}$$

where $|x| = (0, 1)$ and $|y| = (n-1, 2)$. Note that (i, k) corresponds to the ith homological degree and the Configuration space on k many points.

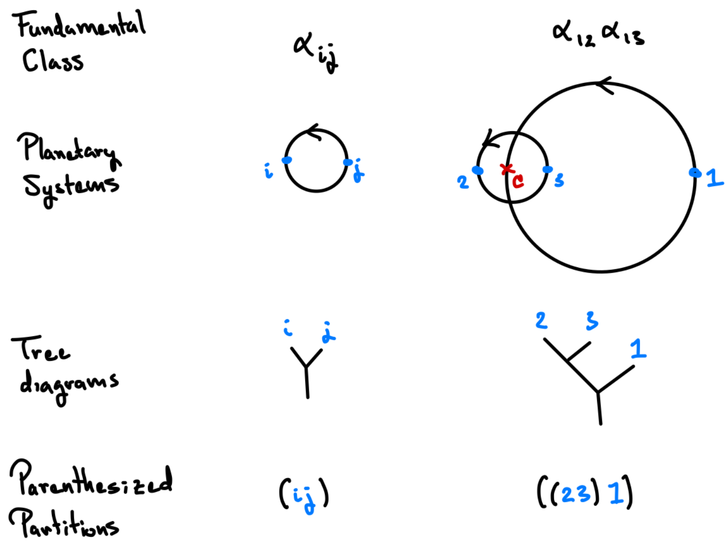

Figure 6. Different formulations of α_{ij}. Note that $\alpha_{ij} \simeq S^1$ and $\alpha_{12}\alpha_{13} \simeq T^2$.

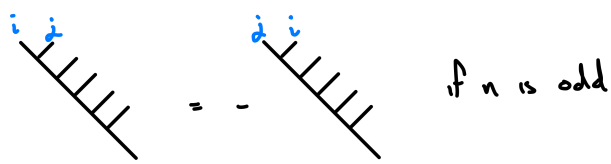

Figure 7. Proof by picture of Corollary 1.6.

2. Lecture 2

Question. What is $H^*(F_k(T^2))$?

Question. What is $H^*(B_k(S_g), \mathbb{F}_p)$ for any $g > 0$?

These are not easy questions. In fact, first question has an answer for $k \leq 7$ which required computer assisted computations. This lecture will go over one approach that allowed us to compute $H^*(F_k(T^2))$. We start off from last lecture with cohomology instead of homology:

$$H^*(B(\mathbb{R}^n; \mathbb{Q}) \cong \begin{cases} P[x] \otimes \Lambda[x] & \text{neven} \\ P[x] & \text{nodd} \end{cases}$$

where the bigrading of x is $|x| = (0, 1)$ and the bigrading of y is $|y| = (n-1, 2)$.

Definition 2.1. *Let X be a space, k be a positive integer, $k = i + j$ and let*

$$s_{ij} : F_k(X) \to F_i(X) \times F_j(X)$$
$$s_{ij} : (x_1, \ldots, x_k) \mapsto ((x_1, \ldots, x_i), (x_{i+1}, \ldots, x_k))$$

The maps s_{ij} are

1. *Equivariant, in that s_{ij} is $(\Sigma_i \times \Sigma_j)$-equivariant.*
2. *Coassociative, in that the following diagram commutes for any positive integers a, b, c*

$$\begin{tikzcd} F_{a+b+c}(X) \arrow[r] \arrow[d] & F_a(X) \times F_{b+c}(X) \arrow[d] \\ F_{a+b}(X) \times F_c(X) \arrow[r] & F_a(X) \times F_b(X) \times F_c(X) \end{tikzcd}$$

3. *Counital, in that the following diagram commutes*

$$\begin{tikzcd} F_k(X) \arrow[r] \arrow[dr] & F_k(X) \times F_0(X) \arrow[d, "\cong"] \\ & F_k(X) \end{tikzcd}$$

4. *Cocommutative, in that the maps s_{ij}, s_{ji} differ by a block permutation:*

$$\begin{tikzcd} F_k(X) \arrow[r, "s_{ij}"] \arrow[dr, "s_{ji}"] & F_i(X) \times F_j(X) \arrow[d, "\cong"] \\ & F_j(X) \times F_i(X) \end{tikzcd}$$

Definition 2.2. *A symmetric sequence is a collection $\mathcal{X} = \{\mathcal{X}(k)\}_{k \geq 0}$ of (graded) modules with an action of Σ_k on $\mathcal{X}(k)$.*

Example. $H^*(F(X))(k) = H^*(F_k(X))$.

Example. A Σ_k-representation V determines a symmetric sequence V concentrated in height k.

Definition 2.3. *The tensor product of $\mathcal{X}$ and $\mathcal{Y}$ is*

$$(\mathcal{X} \otimes \mathcal{Y})(k) = \bigoplus_{i+j=k} Ind_{\Sigma_i \times \Sigma_j}^{\Sigma_k} \mathcal{X}(i) \otimes \mathcal{Y}(j)$$

this is called the Day convolution tensor product.

The Day convolution tensor product has a symmetric monoidal structure.

Exercise. Over a field,

$$H^*(F(X \coprod Y)) \cong H^*(F(X)) \otimes H^*(F(Y))$$

Definition 2.4. *A twisted commutative algebra (TCA) is a commutative monoid in symmetric sequences.*

Exercise. s_{ij} endows $H^*(F(X))$ with a TCA structure.

Exercise. If $\mathcal{A}$ is a TCA, then $\oplus_{k \geq 0} \mathcal{A}(k)^{\Sigma_k}$ and $\oplus_{k \geq 0} \mathcal{A}(k)_{\Sigma_k}$ inherit algebra structures.

The apparent structure of $P[x] \otimes \Lambda[x]$ comes from splitting points apart. We will delve into TCA's to better understand them before returning to Cohomology of $B(X)$.

Example. Given $\mathcal{X}$, the free TCA on $\mathcal{X}$ is

$$\begin{aligned}\mathrm{Sym}(\mathcal{X})(k) &= \bigoplus_{d\geq 0}(\mathcal{X}^{\otimes d})(k)/\Sigma_d,\\ &\cong \bigoplus_{d\geq 0}\bigoplus_{k_1+\cdots+k_d=k} \mathrm{Ind}_{\Sigma_{k_1}\times\Sigma_{k_2}\times\cdots\times\sigma_{k_d}\rtimes\Sigma_d}\bigotimes_{i=1}^{d}\mathcal{X}(k_i)\end{aligned}$$

In particular, if $\mathcal{X} = R$ a ring or field, as a Σ_1-module, the free TCA on one generator is $\mathcal{S}(k) = R$ for all k, the trivial representation.

Definition 2.5. *$\mathcal{S}$ is the trivial representation.*

Note that TCA is the analogue of a polynomial with one generator.

Exercise. Using our description in terms of forests, show that $H^*(F(\mathbb{R}^n))$ is the free TCA on $\mathcal{L}_n(k) = H^{(n-1)(k-1)}(F_k(\mathbb{R}^n))$.

Since $F_1(X) = X$, the unit $1 \in H^0(X)$ determines a homomorphism

$$\mathcal{S} \to H^*(F(X))$$

via this map, $H^*(F(X))$ is a $\mathcal{S}$-module.

Exercise. For any $\mathcal{X}$,

$$(\mathcal{S}\otimes\mathcal{X})(k) = \bigoplus_{l\leq k}\bigoplus_{\{1,\ldots,l\}\hookrightarrow\{1,\ldots,k\}}\mathcal{X}(l)$$

So a $\mathcal{S}$-module is a functor from the category of finite sets of injections. Commonly known as an FI-module.

We now return to the cohomology of $B(X)$.

Theorem 2.6. *(Church) If M is a connected manifold of dimension $n > 1$, then $H^*(F(M))$ is finitely generated over $\mathcal{S}$.*

Theorem 2.7. *(Sam-Snowden (2013), Church-Ellenberg-Farb (2014)) Let M be a finitely generated $\mathcal{S}$-module in characteristic zero.*

(1) The function

$$f \mapsto \dim M(k)$$

is eventually equal to a polynomial.

(2) For every partition λ, the multiplicity of the irreducible symmetric group representation indexed by λ is eventually constant.

Corollary 2.8. *The dimension of $H_i(B_k(M);\mathbb{Q})$ is eventually constant.*

Exercise. Let $\mathcal{X}$ be a symmetric sequence such that

(1) $\mathcal{X}(0) = 0$.
(2) $\mathcal{X}(1) = R$, a ring or a field.
(3) $\mathcal{X}(k)$ is finitely generated in each degree.
(4) $\mathcal{X}(k)$ vanishes in fixed degree for k large.

Then $\mathrm{Sym}(\mathcal{X})$ is finitely generated over $\mathcal{S}$.

In particular, since these apply to $\mathcal{L}_n(k) = H^{(n-1)(k-1)}(F_k(\mathbb{R}^n))$ finite generation follows from $\mathbb{R}^n$.

Theorem 2.9. *(Totaro) Let M be a (for simplicity) oriented manifold of dimension $n > 1$. There is a spectral sequence of TCA's such that*

$$E_2 \cong Sym(H^*(M) \otimes \mathcal{L}_n) \Rightarrow H^*(F(M))$$

Church-Ellenberg-Farb showed that $\mathcal{S}$ is a Noetherian TCA. So finite generation follows from the exercise.

3. Lecture 3

Question. What is the stable multiplicity of any non-trivial representation for any manifold not equal to $\mathbb{R}^n, S^n$?

Definition 3.1. *The Chevalley-Eilenburg complex of a Lie Algebra $\mathfrak{g}$ is $CE(\mathfrak{g}) = \mathrm{Sym}(\mathfrak{g}[1])$, $D(xy) = \pm[x,y]$. This complex calculates Lie Algebra Homology (i.e. Torsion over $U_{\mathfrak{g}}$).*

Theorem 3.2. *(Totaro) The spectral sequence*

$$E^2 = CE(H_c^{-*}(M) \otimes \mathrm{Lie}(R[n-1])) \Rightarrow H_*(F(M))$$

This is in homology, not cohomology. The spectral sequence (SS) from Theorem 3.2 does have higher exponentials, but not at the unordered level.

Theorem 3.3. *(Knudsen) Rationally, the coinvariencts in the SS collapse, so $H_*(B(M;\mathbb{Q})$ is calculated by the complex*

$$\mathrm{Sym}(H_c^{-*}(M;\mathbb{Q})^{\otimes v} \oplus H_c^{-*}(M;\mathbb{Q})^{\otimes [v,v]}),$$
$$D((\alpha \otimes v)(\beta \otimes v)) = \alpha\beta \otimes [v,v]$$

where $|v| = (n,1)$ and $|[v,v]| = (2n-1,2)$. That is, v has cohomological degree n and one point in the Configuration space while $[v,v]$ has cohomological degree $2n-1$ and two points in the Configuration space.

Example. Let's calculate $H_*(B_2(\dot{T}),\mathbb{Q})$ using Theorem 3.3, where $\dot{T}$ is the punctured torus. Let x, y be degree 1 generators of $H_c^{-*}(\dot{T};\mathbb{Q}$ and let their cup product be z. Furthermore, let (i,k) be the ith homology and k the

First, note that x, y are degree 1 generators in cohomology and z is a degree 2 generator in cohomology. Since $H_c^{-*} \cong \tilde{H}^{-*}(T^2;\mathbb{Q})$, then x, y become degree -1 generators in homology and z becomes a degree -2 generator in homology. So for x, y $(2,1) \to (1,1)$ and for z $(2,1) \to (0,1)$. We can now place them in Table 1 in the $(0,1),(1,1)$ spots. Let $\tilde{x}, \tilde{y}, \tilde{z}$ be x, y, z decorated by $[v,v]$, respectively. They gain a degree and a point. Thus, $|\tilde{x}, \tilde{y} = (2,2)$ and $|\tilde{z}| = (1,2)$ are placed in the table in their respective positions. Now, we can multiply these generators within Sym. Degrees add when we multiply, but the number of points in the Configuration space do not change. So $|xy| = (2,2), |zx| = (2,1), |zy| = (2,1)$ and $|z^2| = (2,0)$.

This information is put together in this short exact sequence, where degree 2 maps to degree 1 and degree 1 maps to degree 0.

$$\mathbb{Q}\langle \tilde{x}, \tilde{y}, xy \rangle \to \mathbb{Q}\langle \tilde{z}, xz, xy \rangle \to \mathbb{Q}\langle z^2 \rangle,$$
$$xy \mapsto \tilde{z}$$

2	z^2	$\tilde{z}, zx, zy$	$\tilde{x}, \tilde{y}, xy$
1	z	x, y	0
0	1	0	0
(i,k)	0	1	2

TABLE 1. Table for generators of $\mathrm{Sym}(H_c^{-*}(\dot{T};\mathbb{Q})^{\otimes v} \oplus H_c^{-*}(\dot{T};\mathbb{Q})^{\otimes[v,v]})$, where i is the cohomological degree and k is the number of points in the Configuration space.

Cup product sends $xz \mapsto 0, yz \mapsto 0$, so $xy \mapsto \tilde{z}$. Thus, $\dot{T}$ has 2 generators of degree 2, 2 generators of degree 1 and 1 generator of degree 0.

We can now tell $\mathbb{R}^2 \setminus \{p_1, p_2\}$ and $\dot{T}$, which are homotopic to each other, because $\mathbb{R}^2 \setminus \{p_1, p_2\}$ has two generators of degree 2, but none of degree 1 or 0.

Exercise. Calculate $H_*(B(S^n);\mathbb{Q})$.

Question. How do you calculate $H_*(B(\mathbb{C}P^n);\mathbb{Q})$?

Yet another way that we can study configuration spaces is **Locally**.

Exercise. The topology of $B(M)$ is generated by $\{B(U)\}$ where U ranges over a subset of M homeomorphic to $\coprod_k \mathbb{R}^n$ for some k.

Dugger-Isaksen proved that

$$\mathrm{hocolim}_U(B(U)) \xrightarrow{\sim} B(M)$$

Since $B(\coprod_k \mathbb{R}^n) \cong B(\mathbb{R}^n)^k$, we are reduced (in some sense) to studying $\mathbb{R}^n$.

Question. What structure do configuration spaces carry locally?

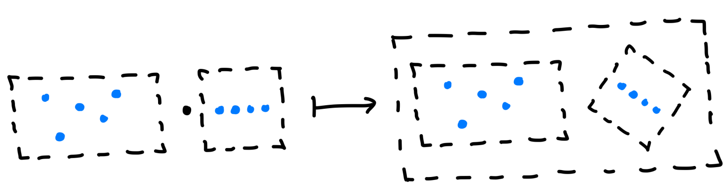

FIGURE 8. Product of Configurations spaces.

Configurations can be multiplied by inserting them into embedded cubes of $\mathbb{R}^n$ as in figure 8. But there are too many choices: how do you embed, how do you order, what size, what shape, etc. We need to keep track of all choices.

$B(\mathbb{R}^n)$ carries k-to-1 operations parametrized by $\mathrm{Emb}(\coprod_k \mathbb{R}^n, \mathbb{R}^n)$, that is, the maps

$$\mathrm{Emb}(\coprod_k \mathbb{R}^n, \mathbb{R}^n) \times B(\mathbb{R}^n)^k \to B(\mathbb{R}^n).$$

Give $B(\mathbb{R}^n)$ the structures of an algebra over the operad Disk_n, in particular over the suboperad $\mathcal{E}_n$ of rectilinear embeddings. In fact, since $\mathcal{E}_n(k)$ knows the center of the squares and their scaling,

$$\mathcal{E}_n(k) \xrightarrow{\sim} F_k(\mathbb{R}^n).$$

So $B(\mathbb{R}^n)$ is a free algebra.

Question. What actually is an $\mathcal{E}_n$-algebra?

Consider the map

$$\mathcal{E}_n(2) \times X^2 \to X$$

Since $\mathcal{E}_n(2) \simeq F_2(\mathbb{R}^n) \simeq S^{n-1}$, this map induces two maps

$$D_*(X)^{\otimes 2} \to H_*(X)$$

of degree 0 and $n-1$, respectively. Because $F_2(\mathbb{R}^n)$ and $F_3(\mathbb{R}^n)$ are connected, the first is commutative and associative. The Arnold/Jacobi relation in $H_*(F_3(\mathbb{R}^n))$ implies that the latter is a Lie bracket. By Ching-Salvatore, there is a map from the shifted spectral Lie operad to $\Sigma^\infty_+(\mathcal{E}_n)$. Then, by universal properties, $\Sigma^\infty_+(B(\mathbb{R}^n))$ forms a spectral higher enveloping algebra.

Theorem 3.4. *(Knudsen) (For simplicity, let M be parallelizable)*

$$\Sigma^\infty(B(M)) \simeq \Sigma_+ \mathrm{Bar}_{\mathrm{Lie}}(\mathrm{Map}_c(M, \mathcal{L}(\Sigma^{n-1}\mathbb{S})))$$

Corollary 3.5. *(Knudsen, partial result of Aoniana-Klein) Stable homotopy types of configuration spaces depends only on the proper homotopy type of M.*

Question. Are Configuration spaces simple homotopy invariants?

Question. Are Configuration spaces homotopy invariants of simply connected manifolds?

This question has been answered for the rational case by Campos-Idrissi-Willwacher.