Algebraic Structures in Topology 2026 Invited Talk: **Telescopic stable homotopy theory**

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Table of Content

1 Invited Talk: Telescopic stable homotopy theory (July 15th, 2026)

Title: Telescopic stable homotopy theory

Abstract: Telescopic stable homotopy theory is an approach to studying the stable homotopy category by decomposing it into atomic periodic pieces, the T(n)-local categories. It is a variant of the approach of chromatic homotopy theory that has the advantage of being richer in information, but is a priori less computable. I will give an overview of some recent advances in understanding the telescopic stable homotopy category, and describe some consequences for this on our overall understanding of homotopy groups of spheres. This talk comes from projects that are joint with Arone, Burklund, Carmeli, Clausen, Hahn, Schlank, and Yanovski.

1.1 Introduction to Chromatic Homotopy Theory

Before reviewing what telescoping homotopy theory is, we will first give an overview of stable homotopy theory. The first question we ask in stable homotopy theory is that:

Question 1.1. What are the stable homotopy groups of spheres (i.e., homotopy groups of the sphere spectrum $\mathbb{S}$)?

Slogan: Doing stable homotopy theory is like doing arithmetic in a homotopy coherent way.

The main player in stable homotopy theory is the category of spectra Sp.

Definition 1.2. Sp is the inverse limit

$$\mathrm{Sp} = \lim(\mathrm{Spc}_\bullet \xrightarrow{\Omega} \mathrm{Spc}_\bullet \xrightarrow{\Omega} \ldots)$$

In other words, a spectrum X is a sequence of spaces $X_1, X_2, \ldots$ such that $X_i \simeq \Omega X_{i+1}$. Sp is symmetric monoidal under the smash product $\otimes$.

Definition 1.3. For a spectrum X, and $i \in \mathbb{Z}$, we define

$$\pi_i(X) = \pi_{n+i}(X_n), n >> 0.$$

Spectra can be thought of as an enlargement of algebra, more precisely, there is a fully-faithful embedding

$$\mathrm{Ab} \hookrightarrow \mathrm{Sp}\,.$$

If we consider $\otimes\mathbb{Z}$, we actually have a map

$$-\otimes\mathbb{Z} : \mathrm{Sp} \to \mathcal{D}(\mathbb{Z}).$$

This admits a right adjoint and we have

$$\begin{tikzcd} \mathrm{Sp} \arrow[r, shift left] & \mathcal{D}(\mathbb{Z}) \simeq \mathrm{Mod}_{\mathrm{Sp}}(\mathbb{Z}) \arrow[l, shift left, "-\otimes\mathbb{Z}"] \end{tikzcd}$$

Example 1.4. We can consider the short exact sequence:

$$\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \cong \bigoplus_{p \text{ prime}} \mathrm{colim}_n\, \mathbb{Z}/p^n.$$

If we view these abelian groups as chain complexes at degree 0, this is also a cofiber sequence in $\mathcal{D}(\mathbb{Z})$.

Let us write down some constructions in spectra.

Definition 1.5. For a fixed prime p, we define

$$\mathbb{S}[p^{-1}] := \operatorname{colim}(\mathbb{S} \xrightarrow{p} \mathbb{S} \xrightarrow{p} \ldots).$$

We can do something similar to consider:

$$\mathbb{S} \to \mathbb{S}[p^{-1}, p \in \text{ primes }].$$

For a number n, we also define

$$\mathbb{S}/n := \operatorname{cofib}(\mathbb{S} \xrightarrow{\cdot n} \mathbb{S}).$$

It is essentially due to Serre's finiteness theorem that:

Corollary 1.6. Serre [Ser53]] $\mathbb{S}[p^{-1}, p \in \text{ primes }] \cong \mathbb{Q}$.

Thus we can consider a similar cofiber sequence

$$\mathbb{S} \to \mathbb{S}[p^{-1}, p \in \text{ primes }] = \mathbb{Q} \to \mathbb{Q}/\mathbb{S} \simeq \bigoplus_{p \text{ primes}} \operatorname{colim} \mathbb{S}/p^n.$$

In ordinary algebra with $\mathbb{Z}$, we are sort of done by this point. However, over the sphere spectrum, we can keep going.

Assumption: For simplicity, we will let p be an odd prime for most of this talk, unless mentioned otherwise.

Definition 1.7. In the 1960's, Adams found a map

$$v_1 : \Sigma^{2p-2}\mathbb{S}/p \to \mathbb{S}/p$$

which has the property that it is **not nilpotent**, so any iterate of this map is **not zero**.

We may now consider doing a similar map

$$\mathbb{S}/p \to \mathbb{S}/p[v_1^{-1}].$$

Taking the cofiber here gives us that

$$\mathbb{S}/p \to \mathbb{S}/p[v_1^{-1}] \to \operatorname{colim}_k \Sigma^{?}\mathbb{S}/(p, v_1^k).$$

Morally speaking, this means that roughly studying the homotopy groups of $\mathbb{S}/p$ breaks up into the last two pieces.

Furthermore, we actually completely understand $\mathbb{S}/p[v_1^{-1}]$, similar to the case for $\mathbb{Q}$. This is because of the following:

Definition 1.8. Let KU be the complex K-theory spectrum. Note this has that

$$\pi_\bullet \mathrm{KU} = \mathbb{Z}[\beta^{\pm 1}], |\beta| = 2.$$

There is a map from
$$\mathbb{S}/p[v_1^{-1}] \to \mathbb{S}/p[v_1^{-1}] \otimes \mathrm{KU}\,.$$
This is not quite an isomorphism, but it is almost. It turns out that there are Adams operations on KU/p which gives an action of the p-adic units $\mathbb{Z}_p^\times$ on KU/p.

Theorem 1.9. There is an equivalence
$$\mathbb{S}/p[v_1^{-1}] \xrightarrow{\sim} (\mathbb{S}/p[v_1^{-1}] \otimes KU/p)^{h\mathbb{Z}_p^\times}\,.$$
For $p = 2$ this is due to Mahowald [Mah81, Mah82], and for $p > 2$ this is due to Miller [Mil81].

We can compute the homotopy groups on the right hand side and get:

Corollary 1.10. We have that
$$\pi_\bullet \mathbb{S}/p[v_1^{-1}] \cong \mathbb{F}_p[v_1^{\pm 1}]\{1, \alpha_1\}.$$

More generally, Hopkins-Smith showed that:

Definition 1.11. There exists a spectrum $V_n = \mathbb{S}/(p^{a_0}, v_1^{a_1}, ..., v_{n-1}^{a_{n-1}})$ and a map
$$v_n^{a_n} : \Sigma^? V_n \to V_n.$$
We call $V_n[v_n^{-1}]$ a telescope of height n.

We can then consider the map
$$V_n \to V_n[v_n^{-1}] \to \operatorname{colim} \Sigma^? V_n/v_n^{?a_n}.$$
This means that we can inductively reduce $\pi_\bullet \mathbb{S}$ to understanding $\pi_\bullet V_n[v_n^{-1}]$.

Furthermore, this decomposition is maximal in the following sense.

Theorem 1.12 (Devinatz-Hopkins-Smith [DHS88], Nilpotence Theorem)**.** $\pi_\bullet V_n/$nilpotents is a subring of the polynomial ring $\mathbb{F}_p[v_n]$.

Question 1.13. What is $\pi_\bullet V_n[v_n^{-1}]$.

We can consider its homotopy groups as follows.

Definition 1.14. For each n (and prime p), the **Morava E-theory** of height n is a spectrum E_n such that
$$\pi_\bullet E_n = (\mathbb{Z}[\omega_{p^\infty - 1}][[v_1, ..., v_{n-1}]][\beta^{\pm 1}])^\wedge_p, |\beta| = 2, |v_i| = 0$$
and $\omega_{p^\infty - 1}$ denotes adjoining all roots of unity coprime to p (this ensures this is the Lubin-Tate theory over an algebraically closed field). There is an action from what is called the (extended) Morava stabilizer group $\mathbb{G}_n$ on E_n.

Example 1.15. When $n = 1$, this is the action of $\mathbb{Z}_p^\times$ on p-completed KU, which does line up with the earlier discussion (albeit we used KU/p).

Conjecture 1.16 (Ravenel [Rav84]). There is an equivalence

$$V_n[v_n^{-1}] \xrightarrow{\sim} E_n^{h\mathbb{G}_n} \otimes V_n[v_n^{-1}].$$

The point is that the right hand side is very **computable in terms of arithmetic geometry**, so we would hope to convert the left side to the right side.

We have noted earlier this is true when $n = 1$. However,

Theorem 1.17 ([BHLS23]). This is false for $n \geq 2$.

1.2 Telescope Conjecture Preliminaries

Before discussing the disproof of the telescope conjecture, we first introduce some notations. We have inclusions

$$\mathrm{Sp}_{K(n)} \subset \mathrm{Sp}_{T(n)} \subset \mathrm{Sp}$$

where:

1. $\mathrm{Sp}_{T(n)}$ is the subcategory generated by $V_n[v_n^{-1}] \otimes X$ under limits (as opposed to colimits).
2. $\mathrm{Sp}_{K(n)}$ is the subcategory generated by $E_n^{h\mathbb{G}_n} \otimes V_n[v_n^{-1}] \otimes X$ under limits (as opposed to colimits).
3. There inclusions in fact admit left adjoints, $\mathrm{Sp} \to \mathrm{Sp}_{T(n)}$ called $T(n)$-localization and $\mathrm{Sp} \to \mathrm{Sp}_{K(n)}$ called $K(n)$-localization.

Proposition 1.18. The telescope conjecture is equivalent to asserting

$$\mathrm{Sp}_{T(n)} = \mathrm{Sp}_{K(n)}.$$

Here is the proof idea.

Definition 1.19. There is an action of $\mathbb{Z}$ on $\mathrm{BP}\langle n\rangle$, and we consider

$$R_n = \mathrm{BP}\langle n\rangle^{h\mathbb{Z}}.$$

We then take the algebraic K-theory $K(R_n)$, and the authors proved that the telescope conjecture is false on $K(R_n)$ for $n \geq 1$.

Theorem 1.20. The natural map $K(R_n)_{T(n+1)} \to K(R_n)_{K(n+1)}$ is not an equivalence for $n \geq 1$.

Let us also briefly explain what algebraic K-theory is, albeit not complete due to the brevity of time.

Definition 1.21. K is a functor $\mathrm{Alg}(\mathrm{Sp}) \to \mathrm{Sp}$. Its K_0-group has a nice interpretation in classical algebra.

1.3 Mechanism of Failure

Let us now explain why the telescope conjecture is false for $n \geq 2$.

The $T(n)$-local sphere spectrum $\mathbb{S}_{T(n)}$ is amenable to the usage of Galois theory, and it admits a Galois extension

$$\mathbb{S}_{T(n)} \to \mathbb{S}_{T(n)}[\omega^{(n)}_{p^k}],$$

where the right side admits an action of $(\mathbb{Z}/p^k)^\times$.

Remark 1.22. When $n = 0$, this literally generalizes the cyclotomic extension

$$\mathbb{Q} \to \mathbb{Q}[\zeta_{p^k}],$$

where the action on the right side is by $(\mathbb{Z}/p^k)^\times$.

Just like ordinary Galois theory, we have that:

Theorem 1.23. $\mathbb{S}_{T(n)} \simeq \mathbb{S}_{T(n)}[\omega^{(n)}_{p^k}]^{h\mathbb{Z}^\times/p^k}$.

It is however not clear from the construction whether the following is an equivalence:

Question 1.24. Is the comparison map

$$\mathbb{S}_{T(n)} \to \mathbb{S}^\wedge_{T(n),cyc} := (\operatorname{colim}_k \mathbb{S}_{T(n)}[\omega^{(n)}_{p^k}])^{h\mathbb{Z}_p^\times}.$$

We call the right hand side the **cyclotomic completion**.

Evidently from here, we have inclusions

$$\mathrm{Sp}_{T(n)} \supset \mathrm{Sp}^\wedge_{T(n),cyc} \supset \mathrm{Sp}_{K(n)}.$$

What the authors actually showed is that

$$\mathrm{Sp}_{T(n)} \neq \mathrm{Sp}^\wedge_{T(n),cyc}$$

Thus this amounts to showing that $K(\mathrm{BP}\langle n\rangle^{h\mathbb{Z}}_{T(n)})_{T(n+1)}$ **is not cyclotomically complete**. The proof proceeds in the following steps

1. By [BMCSY24], it is enough to show that the following is not an equivalence

$$K(\mathrm{BP}\langle n\rangle^{h\mathbb{Z}}_{T(n)})_{T(n+1)} \to K(\mathrm{BP}\langle n\rangle_{T(n)})^{h\mathbb{Z}}_{T(n+1)}.$$

2. The issue is that it is very hard to compute K-theory of anything in general. So we siwtched to a different functor called TC (topological cyclic homology), and there is a natural transformation

$$K \implies \mathrm{TC}.$$

   Furthermore, it is easier to compute homotopy groups of $\mathrm{TC}(-)$.

   Using results of Dundas-Goodwillie-McCarthy [DGM12], Land-Mathew-Meier-Tamme (Chromatic Purity) [LMMT24], and Clausen-Mathew-Naumann-Noel [CMNN22], to get that

$$K(\mathrm{BP}\langle n\rangle_{T(n)})^{h\mathbb{Z}}_{T(n+1)} \xrightarrow{\simeq} \mathrm{TC}(\mathrm{BP}\langle n\rangle)^{h\mathbb{Z}}_{T(n+1)}.$$

It is a bit more tricky on the domain, but one in fact gets a commutative square with vertical isomorphisms:

$$\begin{tikzcd} K(\mathrm{BP}\langle n\rangle^{h\mathbb{Z}}_{T(n)})_{T(n+1)} \arrow[r] \arrow[d, "\simeq"'] & K(\mathrm{BP}\langle n\rangle_{T(n)})^{h\mathbb{Z}}_{T(n+1)} \arrow[d, "\simeq"] \\ \mathrm{TC}(\mathrm{BP}\langle n\rangle^{h\mathbb{Z}})_{T(n+1)} \arrow[r] & \mathrm{TC}(\mathrm{BP}\langle n\rangle)^{h\mathbb{Z}}_{T(n+1)} \end{tikzcd}$$

where the left isomorphism comes from [Lev22].

3. The TC computation is approachable enough that one can show that both sides are not isomorphic.

1.4 Applications of Telescope Conjecture Disproof

We can consider the map

$$\begin{tikzcd} \mathbb{S}_{T(n)} \arrow[r, "\text{unit map}"] \arrow[rr, bend right, "0"'] & \mathbb{S}_{T(n)}[\omega^{(n)}_{p^k}] \arrow[r, "\sigma-1"] \arrow[rr, bend left, "0"] & \mathbb{S}_{T(n)}[\omega^{(n)}_{p^k}] \arrow[r, "\text{trace}"'] & \mathbb{S}_{T(n)} \end{tikzcd}$$

Here σ is a **(later topological) generator** of $(\mathbb{Z}/p^k)^\times$.

Using the theory of Massey products (as the first three and last three are nullhomotopies), this will give a map

$$\Sigma\mathbb{S}_{T(n)} \xrightarrow{\tau_k} \mathbb{S}_{T(n)},$$

which will give **intereting elements in** $\pi_1\mathbb{S}_{T(n)}$.

If we consider the sequence

$$\mathbb{S}_{T(n)} \to V_n[v_n^{-1}] \to K(R_{n-1}) \otimes V_n[v_n^{-1}].$$

Theorem 1.25 (Burklund-Carmeli-Hahn-L.-Schlank-Yanovski)**.** The image of τ_k for $k >> 0$ are in linearly independent in $\pi_\bullet K(R_{n-1}) \otimes V_n[v_n^{-1}]$. This implies that

$$\pi_1 V_n[v_n^{-1}] \text{ is infinitely generated }, n \geq 2.$$

Corollary 1.26. There is an asymptotic lower bound

$$\frac{1}{n}\sum_{i=0}^{n} \mathrm{rank}_{\mathbb{F}_p}((\pi_i\mathbb{S})/p) \gtrsim C\log(n),$$

where C is independent of n.

One similarly gets that

Corollary 1.27 (Arone-L.)**.** For $k \geq 2$, there is an asymptotic lower bound

$$\frac{1}{n}\sum_{i=0}^{n} \mathrm{rank}_{\mathbb{F}_p}(\pi_{k+i}S^k \mod p) \gtrsim C'\log(n),$$

where C is independent of k and n.

Remark 1.28. The speaker believes $\mathrm{im}(\pi_i V_n \to \pi_i V_n[v_n^{-1}])$ is asymptotically $C' \log(n)$.

1.5 A Conceptual Explanation

For $n \geq 1$, we have that in $\mathrm{Sp}_{K(n)}$:

1. The absolute Galois group of $\mathbb{S}_{K(n)}$ is $\mathbb{G}_n$.
2. The separable closure of $\mathbb{S}_{K(n)}$ is E_n, with a group action of $\mathbb{G}_n$.

Theorem 1.29 (Burklund-Clausen-L.). (1) and (2) also hold in $\mathrm{Sp}_{T(n)}$.

Very precisely, this means that we have

$$\mathbb{S}_{T(n)} \to F_1 \to F_2 \to \ldots \operatorname{colim} F_i = E_n,$$

where each F_i admits an action of a finite group G_i. On this path, we have that

$$F_i^{hG_i} = \mathbb{S}_{T(n)},$$

and we also have that

$$\mathbb{S}_{K(n)} \simeq (\operatorname{colim} F_i)^{h\mathbb{G}_n = \lim_{\leftarrow \mathbb{N}} G_i}$$

We end the talk with two open questions.

Open Question 1.30. Is $\mathbb{S}_{K(n)} \simeq \mathbb{S}^{\wedge}_{T(n),cyc}$?

Open Question 1.31. What are the homotopy groups

$$\pi_\bullet \mathbb{S}/(p.v_1)[v_2^{-1}].$$

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