MINI-COURSE: ALGEBRAIC K-THEORY

These are lecture notes for a mini-course given by Teena Gerhardt at the *Algebraic structures in topology II* conference that took place June 5-14, 2024 in San Juan, Puerto Rico. They were typed by Sofía Martínez Alberga and Alex Hsu. Any typos or mistakes are the fault of the scrivers (**not** the speaker).

1. Lecture 1

This first lecture is intended to introduce and motivate the topic and so the notes will address the following questions:

(1) What is *Algebraic K-theory*?
(2) Why should we care about *it*?

A concise answer to the first question is the following: Algebraic K-theory is an invariant of rings. Given a ring A, we associate to A a series of abelian groups

$$K_0(A), K_1(A), K_2(A), \dots.$$

In general, it is not immediately clear what exactly is the algebraic K-theory of a ring measuring or telling us about the ring itself. Initially $K_0(A)$ and $K_1(A)$ can be interpreted in a meaningful way. Additionally, even with these interpretations it is also not clear how the K-theory groups are related.

1.1. K_0 and K_1 of a ring.

The idea to keep in mind in this section is that $K_0(A)$ has to do with projective modules over A and $K_1(A)$ is about matrices with entries in A.

Notation 1.1. Let $P(A)$ denote the monoid of isomorphism classes of finitely generated projective A-modules equipped with the binary operation given by direct sum of A-modules. For an A-module M the corresponding element in $P(A)$ is denoted $[M]$.

Definition 1.2. Define $K_0(A)$ to be the Grothendieck group completion of $P(A)$, i.e. the group obtained by formally adjoining inverses to the monoid $P(A)$.

Remark 1.3. Elements of $K_0(A)$ can be written as formal differences $[M]-[N]$ where M and N are finitely generated projective A-modules.

Example 1.4. For a field F, $P(F) \cong \mathbb{N}$, so we get $K_0(F) \cong \mathbb{Z}$.

Remark 1.5. Why do we not look at **all** projective modules over A? Suppose we allow infinitely generated projective modules. Let A^∞ denote an infinitely generated free module over A and let P be a finitely generated projective A-module. We have that $P \oplus Q \cong A^n$ for some Q and $n \leq 1$, hence

$$\begin{aligned} P \oplus A^\infty &\cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots \\ &\cong (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots \oplus (P \oplus Q) \\ &\cong A^\infty. \end{aligned}$$

Thus $P \oplus A^\infty = A^\infty$, and on K-theory, $[P \oplus A^\infty] = [A^\infty] = [P] + [A^\infty]$ implies $[P] = 0$, which means K_0 of A would be 0.

Now we move on to defining $K_1(A)$.

Notation 1.6. Let $\mathrm{GL}(A)$ denote the infinite general linear group of A, i.e. the colimit of

$$GL_1(A) \subset GL_2(A) \subset GL_3(A) \subset \cdots .$$

Notation 1.7. Let $e_{ij}(a) \in GL_n(A)$ denote the elementary matrix which is the identity matrix except for the i, j entry, namely,

$$e_{ij}(A) = \begin{pmatrix} 1 & 0 & \cdots & & & \\ 0 & 1 & 0 & & & \\ \vdots & & \ddots & a & & \\ & & & \ddots & & \vdots \\ & & & & \ddots & 0 \\ & & & \cdots & 0 & 1 \end{pmatrix}$$

for some $a \in A$.

Let $E_n(A) \subset \mathrm{GL}_n(A)$ denote the subgroup generated by these $e_{ij}(a)$. Let $E(A)$ denote $\cup_{i=1}^{\infty} E_n(A)$.

Definition 1.8 (Whitehead, Bass, Schanuel)**.** Define $K_1(A) := GL(A)^{ab} = GL(A)/E(A)$ where the latter equality follows from Whitehead's lemma.

1.2. **Wall's finiteness obstruction.** In the 1960's, Wall was interested in answering the following question, "When is a space homotopy equivalent to a finite CW complex?" We begin with a definition.

Definition 1.9. A space X is finitely dominated if there exists a finite CW complex Y with maps $f\colon X \to Y, g\colon Y \to X$ and a homotopy $g \circ f \simeq 1_X$.

Now with this definition, Wall's question is refined in the following way: if a space is finitely dominated, does it have the homotopy type of a finite CW complex? Wall gave the answer, “not always,” and in fact, Wall gives a criterion on when this is the case.

Theorem 1.10 (Wall)**.** A finitely dominated space X has a finiteness obstruction $\omega(X)$ such that $\omega(X) = 0$ if and only if X has the homotopy type of a finite CW-complex.

It turns out this obstruction $\omega(X)$ lives in $\tilde{K}_0(\mathbb{Z}[\pi_i X])$, where $\tilde{K}_0$ refers to the reduced 0th K-theory. Note that for a ring A, we have a homomorphism

$$\mathbb{Z} \to K_0(A)$$
$$n \mapsto A^n$$

and $\tilde{K}_0(A)$ is the cokernel of this map, and is called the “reduced $K_0(A)$”.

Corollary 1.11. If X is simply connected and finitely dominated then X has the homotopy type of a finite CW complex.

Proof. We have $\omega(X) \in \tilde{K}_0(\mathbb{Z}[\pi_1 X]) = \tilde{K}_0(Z)$ since the space is simply connected. Using the fact that every finitely generated projective $\mathbb{Z}$-module is free, $\mathbb{Z} \cong K_0(\mathbb{Z})$, therefore $\tilde{K}_0(\mathbb{Z}) = 0$. □

1.3. **The s-cobordism theorem.** (“s” is for “simple homotopy”)

Recall that a cobordism between n-dimensional manifolds M and N is an $(n+1)$-dimensional manifold W such that $\partial W = M \sqcup N$. We call a cobordism an h-cobordism if the inclusion $M \hookrightarrow W$ and $N \hookrightarrow W$ are homotopy equivalences.

The s-cobordism theorem, proven in the 1960’s independently by Mazur, Stallings, and Barden, answers the following question: Given $n \geq 5$ and W an h-cobordism between closed smooth n-manifolds M and N, when is this cobordism trivial, i.e. when is $W \cong M \times [0,1]$? It turns out a complete obstruction lies in a quotient of $K_1(\mathbb{Z}[\pi_1 M])$.

1.4. $K_2(A)$**.** In 1967, Milnor defined $K_2(A)$ by means of the Steinberg group:

Definition 1.12. For $n \geq 3$, the Steinberg group $\mathrm{St}_n(A)$ of a ring $A$is the group with generators $x_{ij}(a)$ for $a \in A$, $i, j \in \{1, \dots, n\}$, $i \neq j$, and relations

$$x_{ij}(a)x_{ij}(b) = x_{ij}(a+b)$$

$$[x_{ij}(a), x_{kl}(b)] = \begin{cases} 1 & j \neq k, i \neq l \\ x_{il}(ab) & j = k, i \neq l \\ x_{ij}(ab) & j \neq k, i = l \end{cases}$$

These relations are known as the Steinberg relations.

As an exercise, one can show that the relations are satisfied by the $e_{ij}(a)$ from earlier. Thus there is a map:

$$\phi\colon \mathrm{St}_n(A) \to E_n(A)$$
$$x_{ij}(a) \mapsto e_{ij}(a)$$

Note that the Steinberg relations for n include those for $n-1$, so

$$\mathrm{St}_{n-1}(A) \hookrightarrow \mathrm{St}_n(A)$$

and let $\mathrm{St}(A) = \varinjlim \mathrm{St}_n(A)$.

Definition 1.13. For a ring A, $K_2(A)$ is defined as the kernel of

$$\phi\colon \mathrm{St}(A) \to E(A).$$

1.5. **Higher K-groups.** It is known that K_0, K_1, and K_2 have relationships at the time they were defined, which led people to believe they were specific cases of a common theory.

Example 1.14. Let A be a ring and I an ideal of A. There is an exact sequence

$$\begin{array}{ccccc} K_2(A,I) & \longrightarrow & K_2(A) & \longrightarrow & K_2(A/I) \\ K_1(A,I) & \longrightarrow & K_1(A) & \longrightarrow & K_1(A/I) \\ K_0(A,I) & \longrightarrow & \cdots & & \end{array}$$

The question then was to define $K_n(R)$ for all $n \geq 0$ such that it agreed with how we previously defined K_0, K_1, and K_2, and extended the relationships between them. This was done by Quillen in the 1970's.

Definition 1.15. For a ring A,

$$K_n(A) = \pi_n(\mathrm{BGL}(A)^+)$$

for $n > 0$. This can be extended to $n = 0$ by forcing it to match with the usual K_0.

Definition 1.16. $\mathrm{BGL}(A)^+$ is a CW-complex with a map $\mathrm{BGL}(A) \to \mathrm{BGL}(A)^+$ such that the map $\mathrm{GL}(A) = \pi_1(\mathrm{BGL}(A)) \to \pi_1(\mathrm{BGL}(A)^+)$ is onto and has kernel $[\mathrm{GL}(A), \mathrm{GL}(A)] = E(A)$, and $H_*(\mathrm{BGL}(A)) \cong H_*(\mathrm{BGL}(A)^+)$.

Indeed, the lower K groups agree (but it is not obvious for $K_2(A)$).

Other modern constructions of K-theory were invented with the goal of extending K-theory to algebraic objects other than rings. When restricted to rings, they agree with Quillen's definition.

Theorem 1.17 (Quillen)**.** $K_n(\mathcal{F}_q) = \begin{cases} \mathbb{Z} & n = 0 \\ Z/(q^i - 1) & n = 2i - 1 \\ 0 & \text{otherwise} \end{cases}$

In 2024, Antieau-Krause-Nikolaus gave an algorithm to compute $K_n(\mathbb{Z}/p^k)$. For $K_n(\mathbb{Z})$ most cases are known, except for about a quarter which are related to the next example.

Since it seems to be so inaccessible, why should we study algebraic K-theory? K-theory has all sorts of applications across mathematics. We conclude with one example from number theory.

Example 1.18 (Vandiver's conjecture, due to Kummer 1849)*.* For a prime p, let K be the maximal real subfield of $\mathbb{Q}[\zeta_p]$. Then p does not divide the class number of K.

This conjecture is still open, and has been computationally verified for all primes $p < 2^{31}$. While the conjecture itself is purely number theoretic, Kurihara proved in the 1990's that Vandiver's conjecture is equivalent to the statement $K_{4i}(\mathbb{Z}) = 0$ for $i > 0$.

2. Lecture 2

Recall from the last time that to a ring A, we can associate to it a series of abelian groups, $K_0(A), K_1(A), \ldots$, called K-theory groups, and Quillen defined the K-theory groups more generally for $q > 0$

$$K_q(A) \cong \pi_q(\mathrm{BGL}(A)^+).$$

We give another example of K-theory's interaction with other branches of mathematics, in particular, as it relates to motivic homotopy theory in algebraic geometry. The table below illustrates the corresponding notions.

Topology:	Algebraic Geometry:
spaces	algebraic varieties
singular cohomology	motivic cohomology
topological K-theory	algebraic K-theory
Atiyah-Hirzebruch spectral sequence	motivic spectral sequence

Note that motivic homotopy theory focuses on applying homotopical techniques to algebraic geometry.

Today we answer the question: how are K-theory groups computed? Our previous lecture concluded with discussing the state of the matter in the 1960's to 1970's. In fact the literature of that time, one would mostly find low dimensional calculations using purely algebraic definitions. Even with Quillen's definition, it turned out it was very hard to compute using the plus construction.

As an example consider $K_q(\mathbb{Z}[x]/(x^m))$, then in 1979 Geller and Roberts computed $K_2(\mathbb{Z}[x]/[x^m], \mathbb{Z}) \cong \mathbb{Z}/2$. So now the the question becomes: how do we get past low dimensional calculations? One approach which is especially active today is called *trace methods*. The idea behind these methods are simple: K-theory is hard to compute, so we approximate it by things which are more computable. The first approximation is Hochschild homology, another invariant of rings.

Let A be a ring and define the *cyclic bar complex* $C_\bullet(A)$ defined as

$$\cdots \to A \otimes A \otimes A \otimes A \to A \otimes A \otimes A \to A \otimes A \to A \to 0$$

with boundary maps

$$\delta_i = \sum_{j=0}^{i} (-1)^j d_j$$

and

$$d_j(a_0 \otimes a_1 \otimes \cdots \otimes a_q) = \begin{cases} a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q & 0 \leq j \leq q \\ a_q a_0 \otimes a_1 \otimes \cdots \otimes a_{q-1} & j > q \end{cases}$$

As an exercise, one can check $\delta_i \circ \delta_{i+1} = 0$. We check it a small case. Let $x \otimes y \otimes z \in A \otimes A \otimes A$, then

$$\begin{aligned} \delta_2 \circ \delta_3(x \otimes y \otimes z) =& \delta_2(xy \otimes z - x \otimes yz + zx \otimes y) \\ =& xyz - zxy - (xyz - yzx) + (zxy - yzx) \\ =& 0 \end{aligned}$$

Definition 2.1. The n-th Hoschshild homology of the ring A is $\mathrm{HH}_n(A) = H_n(C_\bullet(A))$, where $C_\bullet(A)$ is the cyclic bar complex of A.

Notice that

$$\mathrm{HH}_0(A) = H_0(C_\bullet(A)) = A/\{ab - ba\} = A/[A, A]$$

. Additionally it is worth noting that there is also a simplicial perspective to this. The *cyclic bar construction*, denoted $B^{\mathrm{cy}}_\bullet(A)$, is comprised of a simplicial abelian group $B^{\mathrm{cy}}_q(A) = A^{\otimes q+1}$ with face maps d_i, defined above and degeneracies s_i defined as

$$a_0 \otimes \cdots \otimes a_q \mapsto a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes a_{i+1} \otimes \cdots \otimes a_q$$

.

Further, by the Dold-Kan correspondence

$$\mathrm{HH}_n(A) = H_n(C_\bullet(A)) \cong \pi_n(|B_0^{\mathrm{cy}}(A)|).$$

Moreover, the cyclic bar construction also has a *cyclic operator*, i.e. at every simplicial level we have an action by cyclic groups. In particular, on the i-th level $A \otimes \cdots \otimes A$, we have the action of C_i. Thus $B_\bullet^{\mathrm{cy}}(A)$ is a so-called *cyclic object* or *cyclic simplicial object* and its geometric realization has an S^1-action.

There is a trace map relating the q-th K-theory group of A to the q-th Hochschild homology of A called the *Dennis trace*, originally defined by R. Keith Dennis. This map is induced by a simplicial map:

$$N_r(\mathrm{GL}_n(A)) \to B_r^{\mathrm{cy}}(M_n(A)) \to B_r^{\mathrm{cy}}(A).$$

where the first map above, which has as domain the set of r-simplices in the nerve of $\mathrm{GL}_n(A)$, is defined as

$$(g_1, g_2, \ldots, g_r) \to (g_1 g_2 \ldots g_r)^{-1} \otimes g_1 \otimes g_2 \otimes \otimes \cdots \otimes g_r$$

and the second map is the trace. Then we construct

$$\begin{aligned} K_q(A) \cong \pi_q(\mathrm{BGL}(A)^+) \xrightarrow{\text{Hurewicz}} & H_q(\mathrm{BGL}(A)^+) \\ & \cong H_q(\mathrm{BGL}(A)) \\ & \cong H_q(\mathrm{GL}(A)) \end{aligned}$$

While Hochschild homology is relatively computable, it unfortunately turns out to not be a great approximation. This is not too surprising given the lack of topological aspects in its definition. Thankfully, in the 80's, Goodwillie announced that not all is lost. He showed that the Dennis trace lifts through *negative cyclic homology*

$$K_q(A) \to HC_q^-(A) \to \mathrm{HH}_q(A),$$

and rationally $HC_q^-(A)$ is often a good approximation to K-theory. Let us now discuss further negative cyclic homology.

Consider the bicomplex in Figure 1, where the maps are

$$b' = \sum_{i=1}^{n} (-i)^i \mathrm{d}_i,$$

$$t(a_0 \otimes \cdots \otimes a_q) = a_q \otimes a_0 \otimes \cdots \otimes a_{q-1}$$

$$N = 1 + t + t^2 + \cdots + t^n$$

. This has corresponding total complex

$$\mathrm{Tot}(C_{*,*})_n = \otimes_{p+q<n} C_{p,q}$$

with differential $d = d^h + d^v$, where d^h is the differential in the "horizontal direction" and d^v is the differential in the "vertical direction".

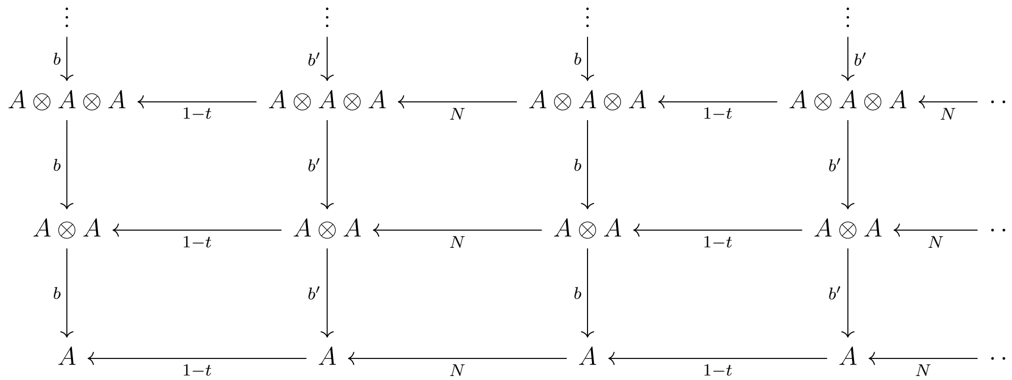

FIGURE 1. Bicomplex in the First Quadrant

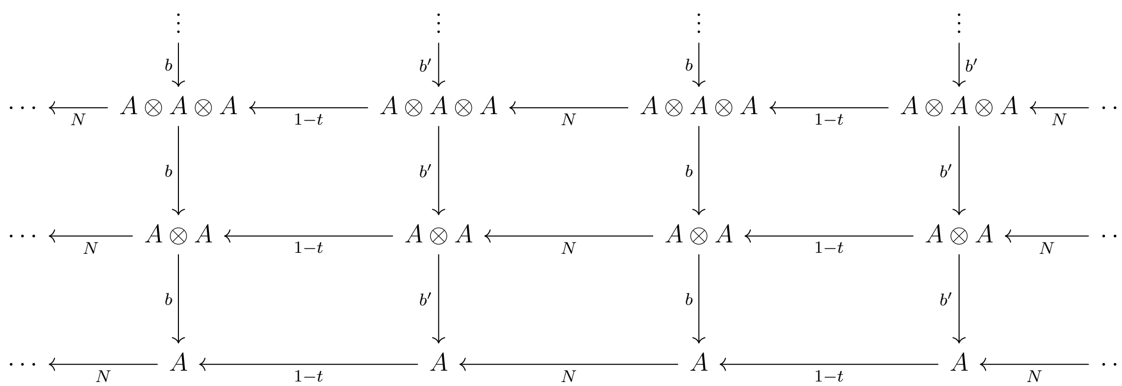

FIGURE 2. Bicomplex Extended to the Second Quadrant

Definition 2.2. The total homology of the bicomplex is cyclic homology $\mathrm{HC}_\bullet(A)$.

If we use periodicity to extend left, as depicted by Figure 2, the homology of the resulting total complex is called periodic homology $\mathrm{HP}_\bullet(A)$. If we only use the columns in negative degrees, as depicted by Figure 3, we get negative cyclic homology $\mathrm{HC}^-(A)$.

Theorem 2.3 (Goodwillie)**.** For A a ring and I a nilpotent ideal, there is an isomorphism

$$K_q(A, I) \otimes \mathbb{Q} \xrightarrow{\cong} \mathrm{HC}_q^-(A \otimes \mathbb{Q}, I \otimes \mathbb{Q}).$$

A theorem of Soulé from 1981 computed the K-theory groups of $\mathbb{Z}[x]/(x^2)$ relative to (x). He deduced that $K_q(\mathbb{Z}[x]/(x^2), (x))$ is finitely generated of rank 1 if q is odd and of rank 0 if q is even. This originally

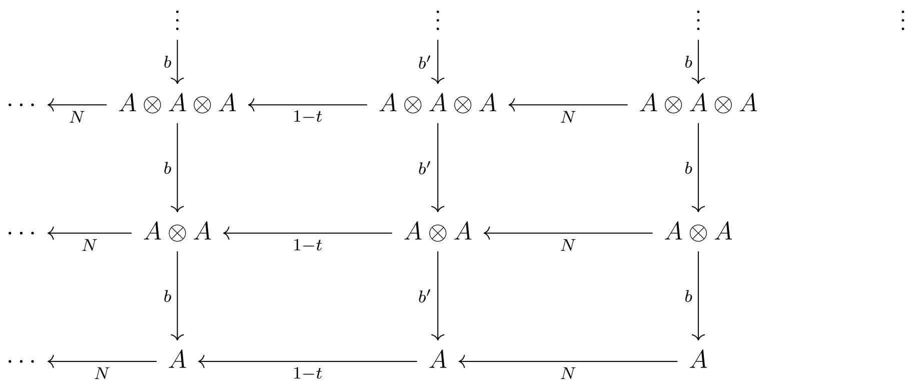

FIGURE 3. Bicomplex Restricted to the Second Quadrant

was not proved using Goodwillie's theorem, but Stasheff used this result to generalize Soulé's theorem.

Theorem 2.4 (Stasheff, 1985)**.** $K_q(\mathbb{Z}[x]/(x^m),(x))$ is finitely generated of rank $m-1$ if q is odd and rank 0 if q is even.

Furthermore, Goodwillie conjectured that there should be topological versions of these theories which captured torsion information, and this idea was dubbed "Brave New Algebra".

The first movement to define topological Hochschild homology was made by Bokstedt. In the table below, we explore the topological analogues for the algebraic tools required to define Hochschild homology.

Algebra:	Topology:
ring, A	ring spectrum, R
tensor product, $\otimes$	smash product, $\wedge$
$\mathbb{Z}$	sphere spectrum, $\mathbb{S}$
$B^{cy}_\bullet(A)$	$B^{cy}_\bullet(R)$

What is exactly is $B^{cy}_\bullet(R)$? It is defined analogously as follows:

$$R\wedge R\wedge R\wedge R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R\wedge R\wedge R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R\wedge R \;\substack{\longrightarrow\\ \longleftarrow\\ \longrightarrow}\; R$$

with face maps

$$d_i = \begin{cases} \mathrm{id}^{\wedge i} \wedge\mu\wedge \mathrm{id}^{q-i-1} & 0\le i<q \\ (\mu\wedge \mathrm{id}^{\wedge q-1})\otimes\tau & i=q \end{cases}$$

and degeneracies

$$s_i = \mathrm{id}^{\wedge(i+1)}\wedge\eta\wedge \mathrm{id}^{(q-i)}\,.$$

We define topological Hochschild homology to be

$$\mathrm{THH}(R) = |B^{\mathrm{cy}}_\bullet(R)|.$$

We will abuse notation and use $\mathrm{THH}(A)$ to denote THH of a **ring** A, where HA is its corresponding Eilenberg-Mac Lane spectrum. These definitions give rise to the *topological Dennis trace*

$$K_q(A) \to \pi_q \,\mathrm{THH}(A),$$

which is what we wanted.

3. Lecture 3

In the previous lecture, we introduced the trace method approach to calculating K-theory groups which approximated K-theory by more computable invariants, namely Hochschild homology. Additionally we discussed a map relating K-theory to Hochschild homology called the Dennis trace map, which lifted it to a more topological version, namely the topological Dennis trace $K_q(A) \to \pi_q \,\mathrm{THH}(A)$.

Classically, the Dennis trace factors as

$$K_q(A) \to \mathrm{HC}^-_q(A) \to \mathrm{HH}_q(A)$$

and rationally, $\mathrm{HC}^-(A)$ can be a good approximation to the K-theory of A. Similarly, a better approximation on the topological side is *topological cyclic homology*, denoted $\mathrm{TC}_q(A)$, originally defined by Bokstedt, Shang, and Madsen in 1993.

In 2018, there was a large advancement in the field by Nikolaus and Scholze who gave a novel approach to computing topological cyclic homology, which we will discuss in this lecture.

Their approach incorporated the *cyclotomic trace map*

$$K_q(A) \to \mathrm{TC}_q(A),$$

and as expected, $\mathrm{TC}(A)$ is often a good approximation to the K-theory of A. In particular, we have the following theorem:

Theorem 3.1 (Dundas-Goodwillie-McCathy)**.** Let A be a ring and I a nilpotent ideal then the map

$$\mathrm{trc} : K_q(A, T) \to \mathrm{TC}_q(A, I)$$

is an isomorphism of groups.

Theorems of this flavor are sometimes referred to as *comparison theorems*. Before continuing we mention some successes of trace methods.

(1) Bokstedt and Madsen, with the help of an additional paper of Tsalidis, computed $K(\mathbb{Z}_p)^\wedge_p$ for $p > 2$. The case for $p = 2$ was computed by Rognes. Another early calculation was due to Hesselholt and Madsen who computed $K(k[x]/(x^m))$ for k a perfect field of characteristic p.
(2) In 2024, Antieau, Krause, and Nikolaus created algorithm to compute $K(\mathbb{Z}/p^k)$. These approaches also work for ring spectra as opposed to just rings, and Ausani-Rognes computed $K(KU)$.
(3) A theorem of Angeltreit, Gerhardt, and Hesselholt used modern approaches to generalize Stasheff's theorem from 1985 (mentioned in the previous lecture). They showed

$$K_q(\mathbb{Z}[x]/(x^m), (x)) \cong \mathbb{Z}^m - 1$$

for q odd, and $|K_q(\mathbb{Z}[x]/(x^m), (x))| = (mq)!(q!)^{m-2}$ for q even.
(4) In 2023, Burklund, Hahn, Levy, and Shlank's disproof of the *telescope conjecture* used trace methods. In simplest terms, the telescope conjecture was a conjecture of Ravenel from 1984, and compared $Sp_{K(n)}$ and $Sp_{T(n)}$. It is known that $K(n)$-local spectra are included in $T(n)$-local spectra, and Ravenel wondered whether it was in fact an equality. Burklund-Hahn-Levy-Shlank constructed counterexamples for all $n \geq 2$. In particular, they gave spectra G_n such that $L_{T(n)}K(G_n)$ are not K_n-local.

We now proceed with the Nikolaus-Scholze approach to topological cyclic homology. Recall that $\mathrm{THH}(A)$ has an S^1-action since it is the realization of a cyclic object.

Definition 3.2. [Nikolaus-Scholze] For a ring A, the topological cyclic homology of A is

$$\mathrm{TC}(A) := \mathrm{Eq}(\mathrm{TC}^-(A) \underset{\varphi}{\overset{can}{\rightrightarrows}} TP(A)^\wedge).$$

We proceed by unpacking the following definitions: $\mathrm{TC}^-(A) := \mathrm{THH}(A)^{hS^1}$ is a topological version of negative cyclic homology. Similarly, $\mathrm{TP}(A) := \mathrm{THH}(A)^{tS^1}$ is topological periodic homology.

Let G be a finite group (or by Greenlees and May, any compact Lie group). Let EG be a contractible space with a free G-action. For X a spectrum with a G-action, we define the homotopy fixed points of X as

$$X^{hG} := F(EG_+, X)^G$$

where $F(EG_+, X)$ denotes the mapping spectrum. These are non-equivariant maps and we consider the fixed points with respect to the conjugation action. We also have the homotopy orbits

$$X_{hG} \coloneqq EG_+ \wedge_G X.$$

Now consider the map

$$EG_+ \to S^0 \to \widetilde{EG}$$

where $\widetilde{EG}$ denotes the cofiber and the first arrow is the map that sends the base point to 0 and everything else to 1. To this we apply the functor $- \wedge F(EG_+, X)$ and we get

$$(EG_+ \wedge F(EG_+, X)) \to S^0 \wedge F(EG_+, X) \to (\widetilde{EG} \wedge F(EG_+, X)).$$

Recall that $S^0 \wedge F(EG_+, X) \cong F(EG_+, X)$. Now we take G fixed points, which yields

$$(EG_+ \wedge F(EG_+, X))^G \to F(EG_+, X)^G \to (\widetilde{EG} \wedge F(EG_+, X))^G.$$

By definition, the middle term is X^{hG} and the first term is isomorphic to X_{hG}, so we have a map $N : X_{hG} \to X^{hG}$, for which we denote the cofiber X^{tG} and call the *Tate fixed points*. Making the appropriate identifications we get the sequence

$$X_{hG} \to X^{hG} \to X^{tG},$$

and this sequence is known as the *Tate spectrum.*

Why is this called so? In algebra, for a finite group G and M a G-module, there is a *norm map*

$$M_G \to M^G$$
$$[x] \mapsto \sum_{g \in G} g \cdot x$$

We could think of this instead as a map

$$H_0(G; M) \to H^0(G; M).$$

Then Tate cohomology is defined as

$$\widehat{H}^i(G; M) \coloneqq \begin{cases} H^i(G; M) & i \geq 1 \\ H_{-i-1}(G; M) & i \leq -2. \end{cases}$$

We extend to $i = 0$ using the sequence

$$0 \to \widehat{H}^{-1}(G; M) \to H_0(G, M) \to H^0(G; M) \to \widehat{H}^0(G; M) \to 0.$$

We have a spectral sequence $\widehat{E}^2_{**} = \widehat{H}^*(G; \pi_* X) \implies \pi_*(X^{tG})$.

Recall definition 3.2 and note that it is the same as

$$Eq(\mathrm{THH}(A)^{hS^1} \underset{\varphi}{\overset{can}{\rightrightarrows}} \mathrm{THH}(A)^{tS^1}).$$

Now we will give the analogous topological definition:

Definition 3.3 (Nikolaus-Scholze)**.** For a ring spectrum R,

$$TC(R) = Eq(\mathrm{THH}(R)^{hS^1} \underset{(\rho_p)^{hS^1}}{\overset{can}{\rightrightarrows}} \pi_p(\varphi_p^* \mathrm{THH}(R)^{tC_p})^{hS^1}).$$

where ρ_p is the isomorphism $\rho_p : S^1 \to S^1/C_p$.

We have

$$\mathrm{THH}(R)^{hS^1} \simeq ((\mathrm{THH}(R)^{hC_p})^{hS^1/C_p} \simeq (\rho_p^* \mathrm{THH}(R)^{hC_p})^{hS^1}$$

then using the canonical map we get

$$can : (\mathrm{THH}(R)^{hC_p})^{hS^1} \to ((\rho_p^* \mathrm{THH}(R))^{tC_p})^{hS^1},$$

which gives one of the maps in the definition.

The other map $(\varphi_p)^{hS^1}$ uses that topological Hochschild homology is a *cyclotomic spectrum*:

Definition 3.4 (Nikolaus-Scholze)**.** A cyclotomic spectrum is a spectrum with an S^1 action and S^1-equivariant maps $\phi_p : X \to X^{tC_p}$ for all p.

Theorem 3.5 (Hesselholt-Madsen, Nikolaus-Scholze)**.** THH(A) is cyclotomic.

This approach made things a lot more computable. For example, we have spectral sequences computing homotopy fixed points and the Tate construction.