Algebraic Structures in Topology 2026 Workshop:
The Chromatic View of Stable Homotopy Theory

Lectures by Ishan Levy
Notes Taken by Mattie Ji

July 10th - 12th, 2026

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Introduction

Algebraic Structures in Topology (2026) is a biennial conference held at San Juan, Puerto Rico featuring minicourses and recent developments in algebraic topology, including the subjects of **homotopy theory**, **low-dimensional geometry**, **topological quantum computing**, and more.

This document contains the notes for the minicourse given by **Ishan Levy** (Institute of Advanced Study) on the subject

The Chromatic View of Stable Homotopy Theory.

The notes were originally live-texed by Mattie Ji, with additional editing works done by Mattie Ji after the minicourse ended.

The course took place from July 10th to July 12th, 2016 and is composed of three 1-hour lectures. The abstracts given by the speaker is as follows:

1. **Lecture 1:** The goal of this talk is to give an overview of the category of spectra and some of its basic properties. We will take the point of view that the category of spectra is a universal place in which algebra can be done.
2. **Lecture 2:** Work of Quillen and Novikov gives a deep connection between the category of spectra and the theory of formal groups. In this talk, I will explain this connection, and discuss some relevant structural results about formal groups, such as the height filtration.
3. **Lecture 3:** I will explain how the height filtration of formal groups can be incarnated at the level of spectra in two ways: as the chromatic filtration as well as a finitary version of it. In doing so I will also explain some of Ravenel's conjectures, and what is still unknown, and its connection to homotopy groups of spheres.

Note: A formal mathematical introduction to ∞-categories would take a lot more than 3 hours. Due to the short nature of the minicourse, some mathematical formalisms were omitted. Instead, more focus were put on motivations and expositions.

The 2026 conference is the 3rd iteration of the conference and is organized by Manuel Rivera (Purdue University), Ralph Kaufmann (Purdue University), Mona Merling (University of Pennsylvania), and Jeremy Miller (Purdue University). The local organizers are Gabriel Montoya Vega (University of Puerto Rico, Río Piedras) and Iván Cardona (University of Puerto Rico, Río Piedras).

If you find any typos or errors in these notes, please feel free to contact the notetaker (Mattie Ji).

1 Lecture 1: Stable Homotopy Theory and Spectra (July 10th, 2026)

This minicourse is about **stable homotopy theory and spectra**, as well as (later) the **chromatic filtration** on spectra. The hope is to try to give people who have maybe never even thought about spectra some ideas about how they work and some sense of chromatic filtrations.

1.1 The "Wrong" Definition for Spectra

We will start off by talking about the "wrong definition" of spectra, or maybe a preliminary one.

Definition 1.1. A **spectrum** $X \in \mathrm{Sp}$ (where Sp denotes the category of spectra) is a sequence of pointed spaces $\Omega^{\infty-i}X$ for $i \in \mathbb{Z}_{\geq 0}$, with the data of a weak homotopy equivalence

$$f_i : \Omega^{\infty-i}(X) \xrightarrow{\sim} \Omega(\Omega^{\infty-i+1}X).$$

Here by a **weak homotopy equivalence**, we mean that the map f_i induces an isomorphism on all homotopy groups, and for a based space Y, ΩY is the space of based loops around the basepoint of Y.

Given two spectra X and Y, we should also define what a map of spectra is.

Definition 1.2. A map of spectra $g : X \to Y$ is the data of maps

$$g_i : \Omega^{\infty-i}X \to \Omega^{\infty-i}Y$$

such that they commute with the weak homotopy equivalences given above. We say h is an equivalence if each g_i is also a weak homotopy equivalence.

Let us look at some examples.

Example 1.3. Let A be an abelian group, there is a spectrum, which we say also call A, such that

$$\Omega^{\infty-i}A := K(A, i),$$

where $K(A, i)$ is the unique CW complex whose π_i is A and $\pi_j = 0$ for $j \neq i$. A is called the **Eilenberg-MacLane spectrum**.

Example 1.4. Another example is the **sphere spectrum** $\mathbb{S} \in \mathrm{Sp}$. Here

$$\Omega^{\infty-i}\mathbb{S} = \mathrm{colim}_{n\geq 0}\,\mathrm{Map}_*(S^n, S^{n+i}).$$

Here $\mathrm{Map}_*(-,-)$ is the space of continuous pointed based maps.

There are some related constructions to spectra.

Definition 1.5. Let $X \in \mathrm{Sp}$, we define

$$\pi_i(X) := \pi_{n+i}\Omega^{\infty-n}X, \text{ for } i >> 0.$$

Example 1.6. Let A be the Eilenberg-MacLane spectrum, then

$$\pi_i(A) = \begin{cases} 0, i \neq 0 \\ A, i = 0 \end{cases}$$

Example 1.7. If we look at the sphere spectrum, we have that

$$\pi_i(\mathbb{S}) = \operatorname{colim}_n \pi_{n+i}(S^n),$$

this is also called the **i-th stable homotopy group of spheres**.

Question 1.8. Asked the speaker, "Why did we state all of these definitions"?

Spectra has appeared in a lot of areas of math, like Floer theory and K-theory. However, from this definition, it is not very clear why they show up naturally. Here we will now give a more motivating context / definition for which spectra show up naturally.

1.2 Rapid Introduction to ∞-categories

Slogan: Spectra is the universal place to do algebra!

The idea the speaker wants to convey is that anytime linear algebra show up, spectra should have a role in it.

In order to do this, we want to change our mathematical framework to a setting where spectra seems more natural. This prompts the usage of the word **infinity categories**. Helpful references include:

1. Higher Algebra [Lur17]
2. Higher Topos Theory [Lur09]
3. Kerodon library [Lur18]
4. Krause-Nikolaus Higher Algebra Lectures on YouTube. https://mathvideos.org/2020/achim-krause-higher-algebra-i/.

Here we want to give some ideas about ∞-categories.

Idea: An ∞-category is really a "topologically enriched category", up to some notion of equivalence.

Definition 1.9 (Informally). A topologically enriched category has objects and morphisms. For x, y objects, $\mathrm{Map}(x, y)$ has the structure of a topological space, and identity, compositions, etc. are continuous. Consequently, $\mathrm{Map}(x, y)$ is called the mapping space.

Now given two objects $x, y \in \mathcal{C}$. We can talk about the idea of homotopy between maps.

Definition 1.10. Let $f : x \to y, g : x \to y$ be maps. A homotopy between maps is a path in $\mathrm{Map}(x, y)$ from f to g. We say f is an **equivalence** if it has an inverse up to homotopy.

Definition 1.11. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor between ∞-categories (i.e., gives continuous maps between each morphism spaces) is an equivalence if:

1. (**fully faithful**): For any two objects $x, y \in \mathcal{C}$, the map
$$\mathcal{C}(x, y) \to \mathcal{D}(F(x), F(y))$$
is a weak homotopy equivalence.
2. (**essentially surjective**): For every $d \in \mathrm{obj}(\mathcal{D})$, d is equivalent to $F(c)$ for some $c \in \mathrm{obj}(\mathcal{C})$. Here by equivalence, we mean the notion in Definition 1.10.

Morally, an ∞-category is really a topologically enriched categories up to the notion of equivalence given above.

Let us look at some examples.

Example 1.12. There is Cat_∞, the ∞-category of small ∞-categories, obtained by inverting the equivalence above.

Example 1.13. There is an ∞-category of spaces Spc, obtained by inverting weak homotopy equivalence from the usual spaces.

Example 1.14. If $\mathcal{C}$ is any ordinary category, it is also an ∞-category, by viewing the hom-sets as discrete spaces. This defines a map
$$\mathrm{Cat} \to \mathrm{Cat}_\infty$$
from the category of categories to the ∞-category of ∞-categories.

Proposition 1.15. The map $\mathrm{Cat} \to \mathrm{Cat}_\infty$ is actually fully-faithful.

Inside Cat_∞, there is the notion of an ∞-Groupoid, which are ∞-categories whose morphsims are all invertible. The induced image above gives
$$\begin{tikzcd} \mathrm{Cat} \arrow[r, hook] & \mathrm{Cat}_\infty \\ \mathrm{Grpd} \arrow[u] \arrow[r] & \infty - \mathrm{Grpd} \arrow[u] \end{tikzcd}.$$

The following is a deep fact whose discrete category analog is not true.

Theorem 1.16. ∞-Grpd is the same as Spc.

Proposition 1.17. The category of groupoids is **Not Equivalent** to the category of sets.

Remark 1.18. Any topological space gives rise to an ∞-category, whose objects are the points. For any two points, the space of pathes between them is the morphism spaces, etc.

Example 1.19. There is also an ∞-category of functors $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$. The maps are somewhat like natural transformations.

There are many parallels between ordinary category and ∞-category, which we explain some as follows.

(1) Many things we enjoy in discrete category theory still carries over.

Usual constructions like: **universal properties, colimits, and limits**, have suitable analogs.

Let us look at some examples.

Definition 1.20 (Initial Object). An object $x \in \mathcal{C}$ in an ∞-category is **initial** if there is a null-homotopy $\mathrm{Map}(x, y) \simeq *$ for all $y \in C$.

An object $y \in \mathcal{C}$ in an ∞-category is **terminal** if there is a null-homotopy $\mathrm{Map}(x, y) \simeq *$ for all $x \in C$.

(2) The role of Set vs. the role of Spc.

In ordinary category theory, we have the following celebrated property of the ordinary category Set.

Theorem 1.21. Let $\mathcal{C}$ be an ordinary category with all colimits, then

$$\underbrace{\mathrm{Fun}^L(\mathrm{Set}, \mathcal{C})}_{\text{Colimit Preserving Functors}} \cong \mathcal{C}.$$

Proof Idea. Evaluate a functor $F \in \mathrm{Fun}^L(\mathrm{Set}, \mathcal{C})$ at the object $* \in \mathrm{Set}$ to obtain the desired isomorphism. ■

In ∞-category, we have an analogous statement given by spaces.

Theorem 1.22. Let $\mathcal{C}$ be an ∞-category with all colimits, then

$$\underbrace{\mathrm{Fun}^L(\mathrm{Spc}, \mathcal{C})}_{\text{Colimit Preserving Functors}} \cong \mathcal{C}.$$

Proof Idea. Evaluate a functor $F \in \mathrm{Fun}^L(\mathrm{Spc}, \mathcal{C})$ at the single point space to obtain the desired isomorphism. ■

In fact, we have the following universal characterzation of Spc.

Theorem 1.23. The ∞-category Spc is freely generated by a point under colimits, as an ∞-category.

1.3 ∞-categories as an Enrichment of Algebra

Let us now try to do **algebras** with ∞-categories!

A prototypical example of an algebraic category is Ab, the ordinary category of **abelian groups**. We can enrich Ab to an ∞-category as follows.

Definition 1.24. We define $\mathcal{D}(\mathrm{Ab})$ as the derived category of Ab whose objects are bounded chain complexes of abelian groups modulo quasi-isomorphisms. This can be made into an ∞-category by

$$\mathcal{D}(\mathrm{Ab}) := \mathrm{Ch}(\mathbb{Z})[\text{quasi-iso}^{-1}],$$

ending quasi-isomorphisms of chain complexes to isomorphisms, and is universal among ∞-categories that do so.

Remark 1.25. Under the enrichment $\mathcal{D}(\mathrm{Ab})$. The map

$$\mathrm{Ab} \hookrightarrow \mathcal{D}(\mathrm{Ab})$$

identifies precisely those objects $C_\bullet$ with $H_i(C_\bullet) = 0$ for $i \neq 0$.

Question 1.26. What properties do $\mathcal{D}(\mathrm{Ab})$ have?

It turns out that $\mathcal{D}(\mathrm{Ab})$ enjoys the following properties:

1. **Pointed:** The initial and terminal objects both exist and are equivalent (it is the zero chain complex).
2. **Stable:** Finite colimits exist, and there is a shift map

   $$\Sigma : \mathcal{D}(\mathrm{Ab}) \to \mathcal{D}(\mathrm{Ab})$$

   that is an equivalence, where ΣX is the **pushout** of

   $$\begin{tikzcd} X \arrow[r] \arrow[d] & 0 \\ 0 & \end{tikzcd}.$$

   The inverse to Σ is denoted

   $$\Omega : \mathcal{D}(\mathrm{Ab}) \to \mathcal{D}(\mathrm{Ab}),$$

   where ΩX is equivalently defined as the **pullback** of

   $$\begin{tikzcd} & 0 \arrow[d] \\ 0 \arrow[r] & X \end{tikzcd}$$

Clearly the definition of stability extends similarly to a pointed ∞-category.

Definition 1.27. A square of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ 0 \arrow[r] & Z \end{tikzcd}$$

is a triangle. It is a **fiber** sequence if it is a pullback. It is a **cofiber** sequence if it is a pushout.

Proposition 1.28 (Stability Characterization)**.** The following are equivalent for a pointed ∞-category $\mathcal{C}$:

1. $\mathcal{C}$ is stable.
2. $\mathcal{C}$ admits finite limits and colimits, a square is a pushout if and only if it is a pullback.
3. $\mathcal{C}$ admits fiber and cofiber sequences, and a triangle is a fiber sequence if and only if it is a cofiber sequence.

Note when we say stable, we usually assume $\mathcal{C}$ is already pointed (it might as well be in the definition).

One consequence of stability is that it gives a convenient place to discuss long exact sequences.

Definition 1.29. For $X, Y \in \mathcal{C}$, we write $[X, Y] := \pi_0 \operatorname{Map}(X, Y)$.

Proposition 1.30. Suppose $\mathcal{C}$ is a stable ∞-category, and $X \to Y \to Z$ is a fiber sequence. For $W \in \mathcal{C}$, the following long exact sequence holds

$$... \to [W, X] \to [W, Y] \to [W, Z] \to [W, \Sigma X] \to ...$$

Furthermore, an analogous long exact sequence holds in the first variable.

Remark 1.31. There is another desirable property of $\mathcal{D}(\mathrm{Ab})$, namely that it is **symmetric monoidal** with tensor product of chain complexes. In fact, $\mathcal{D}(\mathrm{Ab})$ has all limits and colimits.

Let us investigate some more properties of $\mathcal{D}(\mathrm{Ab})$.

Definition 1.32. An object $x \in \mathcal{C}$ is **compact** if $\operatorname{Map}(x, -) : \mathcal{C} \to \mathrm{Spc}$ commutes with all filtered colimits.

For instance, the following is an example of filtered colimit:

$$\operatorname{colim}_{i \in \mathbb{N}} y_i.$$

For a compact object x, we would have

$$\operatorname{Map}(x, \operatorname{colim}_{i \in \mathbb{N}} y_i) \simeq \operatorname{colim}_{i \in \mathbb{N}} \operatorname{Map}(x, y_i).$$

Example 1.33. An abelian group is compact in Ab if and only if it is finitely generated.

A set is compact in Set if and only if it is finite.

Specializing to $\mathcal{D}(\mathrm{Ab})$, we have the following:

Proposition 1.34. Let $\mathbb{Z}$ denote the chain complex with one copy of $\mathbb{Z}$ at the 0-th index, and 0 elsewhere.

$\mathcal{D}(\mathrm{Ab})$ is generated by $\mathbb{Z}$ under Σ, Ω, and colimits. Furthermore, $\mathbb{Z}$ is compact in $\mathcal{D}(\mathrm{Ab})$ and is the unit of $\mathcal{D}(\mathrm{Ab})$ with respect to its tensor product $\otimes$.

1.4 The "Right" Definition for Spectra

Now we are sufficiently motivated to state spectra in the following form.

Definition 1.35. The ∞-category of spectra Sp with the sphere spectrum $\mathbb{S}$ is the universal pair $(\mathrm{Sp}, \mathbb{S})$ such that for any stable ∞-category $\mathcal{D}$ containing all colimits, we have

$$\mathrm{Fun}^L(\mathrm{Sp}, \mathcal{D}) \cong \mathcal{D},$$

by evaluating the functor on the sphere spectrum.

In other words, spectra enjoy the following universal property.

Theorem 1.36. The ∞-category of spectra Sp is the free stable ∞-category generated under colimits by one object (i.e., the sphere spectrum $\mathbb{S}$).

We also have the following parallel description of Sp in contrast to $\mathcal{D}(\mathrm{Ab})$.

Proposition 1.37. Sp is generated by $\mathbb{S}$ under Σ, Ω, and colimits. Furthermore, $\mathbb{S}$ is compact in Sp and is the unit of Sp with respect to a symmetric monoidal operation $\otimes$.

In fact, if we view $\mathbb{Z}$ as the Eilenberg-MacLane spectrum in Sp, we have the following.

Theorem 1.38. There is an equivalence of ∞-categories with $\mathrm{Mod}(\mathbb{Z}) \cong \mathcal{D}(\mathrm{Ab})$.

2 Lecture 2: MU and Formal Group Laws (July 11th, 2026)

Today we will explain the connections between stable homotopy theory and formal groups. Let us review a bit of what we did last lecture.
Recall we were thinking about ∞-categories, which, for our purposes, are categories enriched in topological spaces up to a notion of weak equivalences. We explained:

1. Spc: the ∞-category of spaces under weak equivalences. This had the very nice property that it is freely generated under colimits by a point.

2. Sp: the ∞-category of spectra. This had the very nice property that it is the free stable ∞-category generated by the sphere spectrum under colimits.

2.1 Spectra and Cohomology

There is a functor
$$\Sigma^\infty_+ : \mathrm{Spc} \to \mathrm{Sp}, * \to \mathbb{S}$$
given by sending a point to the sphere spectrum.

To explain this a bit more, the functor above comes from the composition of two functors:

$$\begin{tikzcd}
\mathrm{Spc} \arrow[rr, "\Sigma^\infty_+"] \arrow[dr, "(-)_+"'] & & \mathrm{Sp} \\
& \mathrm{Spc}_* \arrow[ur, "\Sigma^\infty"'] &
\end{tikzcd}$$

where Spc_* denotes the ∞-category of based spaces, under weak based equivalences, and Σ^∞ sends S^0 to $\mathbb{S}$.

These functors described are left adjoints, and they admit right adjoint functors:

$$\begin{tikzcd}
\mathrm{Spc} \arrow[rr, "\Sigma^\infty_+"] \arrow[dr, shift left, "(-)_+"] & & \mathrm{Sp} \arrow[dl, shift left, "\Omega^\infty"] \\
& \mathrm{Spc}_* \arrow[ul, shift left] \arrow[ur, shift left, "\Sigma^\infty"] &
\end{tikzcd}$$

Remark 2.1. A pointed ∞-category is an ∞-category that has an initial and final object which agrees with each other. Spc_* is the free pointed ∞-category generated under colimits by one object.

We will first explain how to get a cohomology theory out of a spectrum.

Definition 2.2. Given a spectrum E, we can get a cohomology theory
$$\mathrm{Spc}^{op} \xrightarrow{E^\bullet} \mathsf{Ab}^{\mathbb{Z}}$$
To define $E^\bullet$, we write
$$E^i(Y) \coloneqq [\Sigma^\infty_+ Y, \Sigma^i E].$$
Here $\mathrm{Ab}^{\mathbb{Z}}$ refers to abelian groups graded by $\mathbb{Z}$, and for spectra X, Y, the $[X, Y]$ is π_0 of the mapping space $\mathrm{Map}(X, Y)$.

Definition 2.3. We also define $E^* = E^*(\mathrm{pt})$, which can be equivalently written as $\pi_{-*}(E)$.

2.2 Complex Oriented Cohomology Theory and Formal Group Laws

Now we will discuss the beginnings of complex oriented cohomology theories. Here we fix R to be a **homotopy ring spectrum** (we will not use ring spectrum in this talk).

Definition 2.4. A **homotopy ring spectrum** R is a spectrum equipped with maps

$$i : \mathbb{S} \to R, \mu : R \otimes R \to R,$$

thought of as the identity and multiplication maps, such that μ is homotopy commutative and homotopy associative, and i is a homotopy identiy under μ. (Here a ring spectrum would include the data of homotopies, we do not use them and only assume they exist).

We now define complex orientability as follows.

Definition 2.5. Let R be a (homotopy) ring spectrum. We say R is **complex orientable** if it is equipped with R-valued Chern classes.

To elaborate, recall Chern classes really come from the cohomologies of $BU(n)$'s. Here we require there to be an isomorphism

$$R^*(BU(n)) \cong R^*(\text{pt})[[c_1^R, ..., c_n^R]], |c_i^R| = 2i.$$

that is compatible with sums.

Example 2.6. When $R = H\mathbb{Z}$, we know that $(H\mathbb{Z})^*(\text{pt}) = \mathbb{Z}$. This is complex oriented. Since $\mathbb{Z}$ has degree 0, this actually recovers the usual cohomology of $BU(n)$, which one may remember is polynomial.

To explain this, observe the natural map

$$\mathbb{Z}[t] \to \mathbb{Z}[[t]],$$

is an isomorphism of graded rings, because at each homogeneous degree, it is the isomorphism from $\mathbb{Z} \to \mathbb{Z}$.

The previous definition appears to be a lot of data in defining complex orientation, but it is actually enough to do the following:

Proposition 2.7. Consider $\Sigma^\infty(\text{BU}(1) = \mathbb{C}P^\infty)$ and the map $\Sigma^\infty(S^2 = \mathbb{C}P^1) = \Sigma^\infty BU(1)$.

A complex orientation is equivalent to the data of a factorization:

$$\begin{tikzcd}
\Sigma^\infty S^2 \arrow[r, "="] \arrow[d] & \Sigma^2\mathbb{S} \arrow[r, "\Sigma^2 i"] & \Sigma^2 R \\
\Sigma^\infty \text{BU}(1) \arrow[urr, dashed] & &
\end{tikzcd}$$

There is another way to make complex orientations as follows.

Proposition 2.8. If $f : R \to R'$ is a map of ring spectrum, and R is complexed oriented, there exists a unique way to make R' complex oriented such that f sends c_1^R to $c_1^{R'}$.

Let us focus a bit more on $R^*(BU(1))$ for R complex oriented. We have that

$$R^*(BU(1)) = R^*[[c_1]], |c_1| = 1$$

For notational ease, let us write $t = c_1$.

Proposition 2.9. Let R be complex oriented. There is a Kunneth formula of the form

$$R^*(BU(1) \times BU(1)) \cong R^*[[t_1, t_2]].$$

Since $BU(1)$ classifies complex line bundles, there is a map

$$\otimes : BU(1) \times BU(1) \to BU(1)$$

that classifies the tensor product of line bundles. Taking cohomology gives a map

$$\phi : R^*[[t]] \to R^*[[t_1, t_2]].$$

This is a map of rings and is determined by what happens to t.

Definition 2.10. We define $F_R(t_1, t_2)$ as the image of t under the map ϕ above. This is called the **formal group law** associated to R.

Let us explain why this is called the **formal group law**. This is not just any random power series, it satisfies certain nice properties.

Since taking tensor product is a commutative and associate operation for line bundles, this gives rigidity on the properties of F_R.

Proposition 2.11. F_R satisfies the following properties:

1. (Identity) $F_R(0, t_2) = t_2$.
2. (Commutativity) $F_R(t_1, t_2) = F_R(t_2, t_1)$.
3. (Associativity) $F_R(x, F_R(y, z)) = F_R(F_R(x, y), F_R(z))$.

It turns out that these axioms imply there exists $g(t) \in R[[t]]$ such that $F_R(t, g(t)) = 0$. It is a feature of the fact that we are working over a power series ring.

The list of properties written above is essentially the definition of an (abstract) formal group law.

Definition 2.12. Let A be a ring. A formal group law (fgl) over A is a power series $F \in A[[t_1, t_2]]$ satisfies the identity, commutativity, and associativity axioms in Proposition 2.11.

There is a bit more structure in the formal group law arising from topology, namely the ring is graded.

Definition 2.13. Let A be a graded ring. A graded formal group law over A is a power series $F \in A[[t_1, t_2]]$ of homogeneous degree 2, with $|t_1| = |t_2| = 2$, satisfiying the axioms in Proposition 2.11.

Remark 2.14. To clarify, in the graded setting, when we write

$$F_R(t_1, t_2) = \sum c_{ij} t_1^i t_2^j,$$

for F_R to be homogeneous of degree 2, we would need that

$$|c_{ij}| = -2i - 2j + 2.$$

Putting our discussions together, there is a functor

$$\mathrm{ComplexOrientedRingSp} \to \text{graded FGL}, \quad R \mapsto F_R \text{ over the graded ring } R^*.$$

It turns out that there are two universal examples for complex oriented ring spectra and graded FGL respectively.

Definition 2.15. There is a graded ring L, called the **Lazard ring**, with the universal (graded) formal group law, with the properties that

$$\mathrm{Hom}_{\text{gradedRings}}(L, A) \cong \{\text{fgls}/A\}.$$

Here is a very non-obvious fact about L.

Theorem 2.16 (Lazard, [Laz55])**.** L is isomorphic to a graded polynomial ring

$$\mathbb{Z}[b_1, i \geq 1], |b_i| = 2i.$$

It turns out one can try to do the same side for complex oriented ring spectra.

Proposition 2.17. There is a universal complex oriented ring spectrum MU (we will construct this later), such that the following two are the same:

1. homotopy classes of ring maps in Spectra $MU \to R$.
2. the set of complex orientations on R.

Remark 2.18. If R is not complex orientable, then there are no rings maps from MU to R.

We will explain how to construct MU. The space $BU(n)$ is the universal space for complex rank n bundles, and it has a canonical bundle

$$\gamma_n \to BU(n).$$

One can consider the $\mathrm{Th}(\gamma_n)$ (Thom space)

$$\mathrm{Th}(\gamma_n) = D(\gamma_n)/S(\gamma_n),$$

where $D(\gamma_n)$ is the unit disk bundle and $S(\gamma_n)$ is the unit sphere bundle.

The natural map $BU(n-1) \to BU(n)$ pulls γ_n back to $\gamma_{n-1} \oplus \epsilon^1$. The functoriality of Thom space, gives a map

$$\Sigma^2 \mathrm{Th}(\gamma_{n-1}) \to \mathrm{Th}(\gamma_n). \quad (\dagger)$$

Definition 2.19. MU is given by the following colimit in spectra

$$\mathrm{colim}_{n \geq 0} \Sigma^{-2n} \Sigma^\infty \mathrm{Th}(\gamma_n)$$

where the transition maps are given as in $(\dagger)$.

Remark 2.20. Milnor computed the homotopy groups of MU in [Mil60] as $\mathbb{Z}[b_i, i \geq 1]$ with $|b_i| = 2$, using the Adams spectral sequence.

2.3 Quillen's Theorem and Moduli Stack of Formal Groups

Now we will explain Quillen's celebrated theorem.

Theorem 2.21 ([Qui]). The classifying map from $L \to \mathrm{MU}^*$ is an isomorphism. This identifies the formal group law of MU^* as the universal formal group law.

Remark 2.22. Quillen's proof expanded upon Milnor's Adams spectral sequence calculations. Quillen later also had an alternative proof of this using power operations.

This is a remarkable theorem that the two sides of stable homotopy theory and algebra have an interesting connection. This in fact goes a bit further.

$MU \otimes MU$ is a ring spectrum, and it has at least 2 complex-orientations by the following two maps:

$$MU \to MU \otimes \mathbb{S} \xrightarrow{id \otimes unit} MU \otimes MU,$$

$$MU \to \mathbb{S} \otimes MU \xrightarrow{unit \otimes id} MU \otimes MU,$$

which we will call F^L and F^R. Now consider the map

$$(MU \otimes MU)^*(BU(1)) \to (MU \otimes MU)^*(BU(1) \times BU(1)),$$

this is just a well-defined map and gives both F^L and F^R from the ring map.

This will give the following:

Theorem 2.23. There exists $f \in MU^*(MU)[[t]]$ such that

$$F^R(f(t_1), f(t_2)) = f(F^L(t_1, t_2)),$$

i.e., the two formal group laws differ by a power series f. Furthermore, $f = t + O(t^2)$ (i.e., its constant term is 0).

Definition 2.24. Let A be a graded ring and F, G be two graded formal group laws. A strict isomorphism $f : F \to G$ is a power series $f \in A[[t]]$ such that

$$F(f(t_1), f(t_2)) = f(G(t_1, t_2)),$$

and $f = t + O(t^2)$.

The analog of Quillen's theorem is also true in this case.

Theorem 2.25. There is a universal graded ring with two formal group laws F_1, F_2 with strict isomorphism $f : F_1 \to F_2$. This ring is realized by $(MU \otimes MU)^*$.

This analogy goes further.

Theorem 2.26. For $n \geq 1$, the ring $(MU^{\otimes n})^*$ is the universal graded ring with n formal group laws, $F_1, \ldots, F_n$, with a chain of strict isomorphisms

$$F_1 \xrightarrow{f_1} F_2 \xrightarrow{f_2} F_3 \ldots \to F_n.$$

Let us now consider the diagram

$$\begin{tikzcd} \mathbb{S} \arrow[r, "\text{unit}"] & MU \arrow[r, shift left=3] \arrow[r, shift right=3] & MU \otimes MU \arrow[l, "\mu"'] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & MU \otimes MU \otimes MU \arrow[l, shift right=6] \arrow[l, shift left=6] & \ldots \end{tikzcd}$$

This is some sequence of ring spectra. Let us focus on the subsequence

$$\begin{tikzcd} MU \arrow[r, shift left=3] \arrow[r, shift right=3] & MU \otimes MU \arrow[l, "\mu"'] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & MU \otimes MU \otimes MU \arrow[l, shift right=6] \arrow[l, shift left=6] & \ldots \end{tikzcd}$$

and take the cohomology on a point.

This gives a sequence

$$\begin{tikzcd} MU^* \arrow[r, shift left=3] \arrow[r, shift right=3] & (MU \otimes MU)^* \arrow[l, "\mu"'] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & (MU \otimes MU \otimes MU)^* \arrow[l, shift right=6] \arrow[l, shift left=6] & \ldots \end{tikzcd} .$$

When considering maps into R, this gives a diagram of the form

$$\begin{tikzcd} \text{fgls over } R \arrow[r, "\text{id}"] & \text{2 fgls over } R\text{, with strict iso} \arrow[l, shift right=4, "\text{send } F_1"'] \arrow[l, shift left=4, "\text{send } F_1"] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & \ldots \arrow[l, shift right=6] \arrow[l, shift left=6] \end{tikzcd}$$

Observation: This diagram represents (in some sense) the groupoid of formal group laws and isomorphisms between them. In this perspective.

Definition 2.27. A (graded) formal group is an object in this groupoid of fgls with strict isomorphisms.

We can think of then a functor

$$\text{graded rings} \to \text{groupoids} \subsetneq \text{Spc}$$

$$R \to \text{groupoid of formal groups over } R.$$

Definition 2.28. This groupoid is the moduli stack of formal groups, which we denote M_{fg}.

Remark 2.29. M_{fg} is not an Artin stack, but it is a pro-Artin stack.

Definition 2.30. We define $\mathrm{Fun}(\mathrm{gradedRings}, Sp)$ as $\mathrm{PreStacks}$. The map

$$\mathrm{gradedRing}^{op} \to \mathrm{PreStacks}, \quad R \to \mathrm{Hom}(R, -).$$

In particular, we have that $\mathrm{Hom}(R, M_{fg})$ as the groupoid of formal groups over R.

3 Lecture 3: The Chromatic Filtration (July 12th, 2026)

Today we will talk about the **chromatic filtration**. Last time, we talked about the connections between formal groups and spectra, today we will see how this connection is reflected within the internal structure of Sp.

3.1 The p-local Sphere Spectrum

To motivate this, let us look at one of the fundamental problems in stable homotopy theory.

Question 3.1. Let $\mathbb{S} \in \mathrm{Sp}$, what are the homotopy groups $\pi_\bullet \mathbb{S}$.

Here are some basic facts about this:

1. For $k < 0$, $\pi_k(\mathbb{S}) = 0$.
2. For $k = 0$, $\pi_0(\mathbb{S}) = \mathbb{Z}$.
3. For $k > 0$, $\pi_k(\mathbb{S})$ is a finite abelian group.

In the first lecture, we mentioned that morally one should think about Sp as the derived category of the $\mathbb{Z}$, but with $\mathbb{Z}$ replaced by $\mathbb{S}$.

Let us look at the positive homotopy groups of $\mathbb{S}$. Well, since they are finite abelian groups, we can **apply the classification theorem of finite abelian groups** to break them into p-primary components for each prime p.

Definition 3.2. $(\pi_\bullet \mathbb{S})_{(p)}$ denotes the p-primary part of $\pi_\bullet \mathbb{S}$.

What is intriguing is that this can actually be realized as the homotopy groups of a spectrum.

Definition 3.3. The p-local sphere spectrum is

$$\mathbb{S}_{(p)} \coloneqq \mathrm{colim}(\mathbb{S} \xrightarrow{q_1} \mathbb{S} \xrightarrow{q_2} \mathbb{S} \ldots)$$

where $q_1, q_2, \ldots$ enumerate the sequence of prime numbers that are not p itself. In other words, we invert all primes other than p itself.

Proposition 3.4. The homotopy groups of $\mathbb{S}_{(p)}$ are $(\pi_\bullet \mathbb{S})_{(p)}$.

Thus the problem for understanding the stable homotopy groups of spheres becomes a question about $\mathbb{S}_{(p)}$ at each prime p.

The discussions so far could also have bbeen done on a **categorical level**. We are spelling this out because chromatic filtration is going to be a fancier version of what we did above.

Definition 3.5. We define the full subcategory $\mathrm{Sp}_{(p)} \subset \mathrm{Sp}$ as spectra whose homotopy groups are already p-localized. The inclusion

$$\mathrm{Sp}_{(p)} \subseteq \mathrm{Sp}$$

has a **left adjoint** called p-localization $(-)_{(p)}$.

Definition 3.6. Let M be a spectrum, we can form $M[p^{-1}]$ by

$$M[p^{-1}] \coloneqq \operatorname{colim}(M \xrightarrow{\cdot p} M \xrightarrow{\cdot p} M \xrightarrow{\cdot} M \dots)$$

Here p is given by the sum of the identity map p times.

Remark 3.7. The p-local sphere spectrum $\mathbb{S}_{(p)}$ has a map

$$\mathbb{S}_{(p)} \to \mathbb{S}_{(p)}[p^{-1}].$$

The target is effectively rational as all of its primes are inverted, one actually has that

$$\mathbb{S}_{(p)}[p^{-1}] = \mathbb{Q}$$

is the Eilenberg-MacLane spectrum at $\mathbb{Q}$.

Recall there is a notion of fiber and cofiber sequence in spectra. Here consider the fiber

$$\text{fib} \to \mathbb{S}_{(p)} \to \mathbb{Q}.$$

The homotopy groups of this fiber sequence fit in a long exact sequence, and one actually has that

$$\text{fib} = \Sigma^{-1} \operatorname{colim}_n \mathbb{S}/p^n.$$

Here $\mathbb{S}/p^n \coloneqq \operatorname{cofib}(\mathbb{S} \xrightarrow{\times p^n} \mathbb{S})$.

The chromatic filtration says that one can iterate this process in the remark further.

3.2 Formal Groups and Quasicoherent Sheaves

Recall from last time, we have a **moduli of formal groups**.

Definition 3.8. There is a functor from

$$\text{gradedRings} \to \text{groupoids} \subseteq \text{Spc}$$

by sending a graded ring A to the groupoid of formal group laws over A and strict isomorphisms of formal group laws.

To briefly recall what this means, a formal group law is a power series $F(t_1, t_2)$ satisfiying properties, and a strict isomorphism is a power series $f(t)$ of the form $f(t) = t + O(t^2)$ that acts on F by

$$f \cdot F \mapsto f(F(f^{-1}(t_1), f^{-1}(t_2))).$$

The reason why we have this name is that we can think of it as a geometric object.

Definition 3.9. There is a fully faithful embedding

$$\text{gradedRings}^{op} \to \text{Fun}(\text{gradedRing}, \text{Spc}),$$
$$A \mapsto \text{Spec}(A).$$

Here $\text{Spec}(A)(B) \coloneqq \text{Hom}_{gradedRing}(A, B)$.

The target $\mathrm{Fun}(\mathrm{gradedRing}, \mathrm{Spc})$ is called prestacks.

Given a prestack X, we can define **quasicoherent sheaves** over X as

$$\mathrm{QCoh}(X) \coloneqq \lim_{\mathrm{Spec}\, A \to X} \mathrm{Mod}(A),$$

where $\mathrm{Mod}(A)$ denotes graded A-modules.

Let us look at some examples.

Example 3.10. If we take $\mathrm{QCoh}(\mathrm{Spec}\, A)$. This is just $\mathrm{Mod}(A)$, i.e.,

$$\mathrm{QCoh}(\mathrm{Spec}\, A) = \mathrm{Mod}(A).$$

If we consider $\mathrm{QCoh}(M_{fg})$, it is given by the following data

- An assignment $(A, F_A) \to M_{F_A}$ to some module.
- Whenever there is a map $f : A \to B$ sending $f : F_A \to F_B$. There is a compatible isomorphism of the form
$$M_{F_A} \otimes_A B \cong M_{F_B}.$$

This seems like a lot of data, but by Quillen's theorem, one has the following.

Corollary 3.11 (Informal). $\mathrm{QCoh}(M_{fg})$ is the same as the data of MU^*-modules M and an identification of $(MU^* \otimes MU)^* \otimes_{MU^* M}$ coming from η_L, η_R (the two formal group laws), plus certain compatibilities.

Definition 3.12. Let $X \in \mathrm{Sp}$, we define

$$\mathrm{MU}_*(X) \coloneqq \pi_* X \otimes \mathrm{MU}\,.$$

This is an MU_*-module, and the two maps from the left and right inclusion give

$$\pi_* X \otimes \mathrm{MU} \rightrightarrows X \otimes \mathrm{MU} \otimes MU.$$

This refines to an object of $\mathrm{QCoh}(M_{fg})$, which we will call $\mathcal{F}_X$.

This quasicoherent sheaf $\mathcal{F}_X$ has a lot of useful applications. For example, there is a **spectral sequence** called the **Adams-Novikov** spectral sequence [Nov67]. For a spectrum X, its has signature

$$E_2^{p,q} = \mathrm{Ext}^{p,q}_{\mathrm{QCoh}(M_{fg})}(\mathcal{F}_{\mathbb{S}}, \mathcal{F}_X) \implies \pi_\bullet X.$$

This **converges** under nice conditions, for example when $\pi_k(X) = 0$ for $k << 0$. For example, it **converges for the sphere spectrum**.

3.3 The Chromatic Filtration of M_{fg}

Let us now discuss the chromatic filtration. To do this, we need to discuss the notion of chromatic height.

Definition 3.13. Let A be a graded ring and F, G be two formal group laws over A. A **homomorphism** $f : F \to G$ is a power series $f \in A[[t]]$ (without the requirement on $f(t) = t + O(t^2)$), such that

$$f(F(-,-)) = G(f(-), f(-)).$$

Proposition 3.14. When A is a characteristic p ring and $f \in A[[x]]$. Then either $f = 0$ or f is of the form $g(x^{p^h})$ for $h \in \mathbb{Z}_{\geq 0}$ such that

$$g = ax + ..., a \neq 0.$$

Proof Idea. We can think of Spec $A[[x]]$ as a formally completed affine line, and a formal group law is a group law along this line. We can morally think about this as a Lie group which has a Lie algebra, which is exactly the **invariant 1-form** ω_F. The map f preserves the invariant 1-form, thus

$$f^*\omega_F = a\omega_F, \text{ for } a \in A.$$

One can check that

$$\omega_F = dx + O(x).$$

If a is non-zero (i.e., the first deribative of f at the identity is non-zero), then h is just 0. If $a = 0$, then $\frac{df}{dx} \equiv 0$. In this case, since we are in characteristic p, this would imply that $f = g'(x^p)$.

This can be iterated to get the desired g. ■

Definition 3.15. Let F be a formal group law, we write $F(x, y) = x +_F y$, as a formal notation. We also write the p-series of F as:

$$[p]_F(x) \coloneqq \underbrace{x +_F x +_F ...x}_{p \text{ times}}.$$

Observation: $[p]_F(x)$ is an endomorphism of F, so by Proposition 3.14, we have that

$$[p]_F(x) = g(x^{p^h}).$$

In this case, we say F has height $\geq h$. Furthermore F has height $= h$ if a is a unit in the description of Proposition 3.14.

Remark 3.16. From this definition, it is clear that the height is an invariant of the underlying formal group, not just the formal group law.

The following is an amazing fact.

Proposition 3.17. Over an algebraically closed field k, height **completely classifies formal groups** up to isomorphism.

This height h is between $\{0, 1, ..., \infty\}$. In characteristic 0 only $h = 0$ can happen. The case for $h > 0$ appears only in positive characteristics.

One way to interpret this is that it is giving us a picture on the geometric points of the moduli stack of formal groups. If we think of M_{fg} as some space and focus on prime (p), then pictorially we have:

$$(M_{fg})_{(p)} \qquad \bullet_0 \qquad \bullet_1 \qquad \bullet_2 \qquad \bullet_3 \qquad \cdots \qquad \bullet_\infty$$

There is in fact a topology coming from this where 1 is in the closure of 0, 2 is in the closure of 1, and so on.

Let us be more explicit of what the closure means.

Definition 3.18. The **chromatic filtration** is the sequence:

$$(M_{fg})_{(p)} \supset M_{fg}^{ht\geq 1} \supset M_{fg}^{ht\geq 2} \supset \ldots$$

3.4 The Chromatic Filtration of Spectra

We will now explain how the chromatic filtration is also reflected in spectra.

Definition 3.19. A subcategory of Sp is **thick** if it is closed under pushouts, $\Sigma^{\pm 1}$, and summands.

Definition 3.20. A spectrum $X \in \mathrm{Sp}$ is **finite** if it is in the thick subcategory generated by the sphere spectrum. We denote Sp^{ω} as the category of finite spectra.

The following theorem connects the chromatic filtration to a statement on the stable homotopy category.

Theorem 3.21 (Thick Subcategory Theorem, by Hopkins-Smith [HS98])**.** There exists a bijection between:

1. The set $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ (thought of as set of heights).
2. The **thick subcategories** of finite p-local spectra.

The bijection is given by sending

$$n \mapsto \mathcal{C}_n,$$

where $\mathcal{C}_n$ is the subcategory of $X \in \mathrm{Sp}_{(p)}^{\omega}$ such that $\mathcal{F}_X \in \mathrm{QCoh}_{fg}$ is supported on $\mathcal{M}_{fg}^{ht\geq n}$. $\mathcal{C}_n$ is called the subcategory of spectra of **type** $\geq n$. If a spectrum is in $\mathcal{C}_n$ but not $\mathcal{C}_{n+1}$, it has type exactly n.

Example 3.22. $\mathbb{S}/p$ is an example of a type 1 spectrum.

This theorem can be morally thought of as a spectral version of the classification of finite abelian groups.

The theorem also goes further. Now we will talk about the nilpotence and periodicity theorem.

Theorem 3.23 (Devinatz-Hopkins-Smith [DHS88], and also [HS98])**.** Suppose $X \in \mathrm{Sp}_{(p)}^{\omega}$ and the endomorphism ring $\mathrm{End}_{\bullet}(X) := [\Sigma^{\bullet} X, X]$, which is a graded ring under composition.

Consider the ring map

$$\phi : \mathrm{End}_{\bullet}(X) \to \mathrm{End}_{\mathrm{QCoh}(M_{fg})}(\mathcal{F}_X).$$

Then the **kernel of** ϕ is nilpotent. Furthermore, ϕ is almost surjective in the following sense, the induced map on the center:

$$Z(\mathrm{End}_{\bullet}(X)) \to Z(\mathrm{End}_{\mathrm{QCoh}(M_{fg})}(\mathcal{F}_X))$$

is **surjective up to taking powers**.

We want to explain one more thing about the chromatic filtration.

Definition 3.24. Let F be a formal group law over A of height $\geq n$. Since F has height $\geq n$, we have

$$[p]_F(x) = v_n x^{p^n} + O(...).$$

One can check this number v_n is **an invariant of the formal group**.

This means that the number v_n gives a global section of $M_{fg}^{ht \geq n}$.

Example 3.25. It turns out $\mathcal{F}_{\mathbb{S}/p}$ is exactly $\mathcal{O}_{M_{fg}^{ht\geq 1}}$. The map v_1 thus arose to be a map

$$\Sigma^? \mathbb{S}/p \xrightarrow{v_1^s} \mathbb{S}/p$$

for some k.

It turns out that if $p > 2$, this has the shape

$$\Sigma^{2p-2}\mathbb{S}/p \xrightarrow{v_1} \mathbb{S}/p.$$

We can also consider the fiber sequence

$$\text{fib} \to \mathbb{S}/p \to \mathbb{S}/p[v_1^{-1}].$$

This fiber has the description of the form

$$\operatorname{colim}_n \Sigma^? \mathbb{S}/(p, v_1^k).$$

It also turns out that

$$\mathbb{S}/p[v_1]^{-1} = \mathbb{F}_p[v_1^{\pm 1}]\{1, \alpha\}.$$

Remark: Much of the discussions ehre was origianlly due to Haynes Miller and Mark Mahowald.

Remark 3.26. $\mathbb{S}/(p, v_1)$ is an example of a type 2 spectrum.

We can extend the v_1 story further to consider v_2. In this case we have a map

$$\Sigma^? \mathbb{S}/(p, v_1 0 \xrightarrow{v_2} \mathbb{S}/(p, v_1).$$

This reduces the study of $\pi_\bullet \mathbb{S}/(p, v_1)$ to the pieces

$$\pi_\bullet \mathbb{S}/(p, v_1, v_2^?) \text{ and } \pi_\bullet(\mathbb{S}/(p, v_1))[v_2^{-1}].$$

Remark 3.27. $\mathbb{S}/(p, v_1, v_2^?)$ has type 3.

In general, we can consider

$$X := \mathbb{S}/(p^{a_0}, v_1^{a_1}, ..., v_{n-1}^{a_{n-1}}), \text{ which has type } n,$$

and we can try to understand $X[v_n^{-1}]$.

Definition 3.28. The spectrum $X[v_n^{-1}]$ is called the **telescope of height** n.

These telescopes are thought of as basic building blocks. It turns out that:

Proposition 3.29. Suppose Y has type $n \geq 1$, then

$$Z(\mathrm{End}(\mathcal{F}_Y)) \text{ mod nilpotence}$$

is a non-trivial subring of the polynomial ring $\mathbb{F}_p[v_n]$.

What this proposition is saying is that $X[v_n^{-1}]$ are really irreducible pieces.

A big question in the field is that:

Question 3.30. How can we understand these telescopes? How can we compute the homotopy group $\pi_\bullet X[v_n^{\pm 1}]$?

One method to try to understand this is to use the Adams-Novikov spectral sequence (ANSS). It turns out however that the ANSS converges to what is called

$$\pi_\bullet L_{K(n)} X[v_n^{\pm 1}],$$

where $L_{K(n)}$ means localization at $K(n)$.

Conjecture 3.31 (Telescope Conjecture by Ravenel [Rav84])**.** $X[v_n^{\pm 1}]$ is equivalent to $L_{K(n)} X[v_n^{\pm 1}]$.

If this is true, then it says that we can understand these basic building blocks using the homological algebra of quasi-coherent sheaves.

Theorem 3.32. This is true for $n = 1$. For $p = 2$, this is due to . For $p = 2$, this is due to Mahowald in [Mah81, Mah82]. For $p > 2$, this is due to Miller in [Mil81].

In recent joint work of the speaker:

Theorem 3.33 (Burkland-Hahn-L.-Schlank [BHLS23])**.** For $n \geq 2$, this is false. In fact, $\pi_\bullet L_{K(n)} X[v_n^{\pm 1}]$ is finite in each degree.

Remark 3.34. In fact, when we look at, for $p > 3$

$$\pi_\bullet \mathbb{S}/(p, v_1)[v_2^{-1}] \to \pi_\bullet L_{K(2)} \mathbb{S}/(p, v_1)[v_2^{-1}].$$

The right side is 12-dimensional vector space over $\mathbb{F}_p[v_2^{\pm 1}]$. However, the left side, even after taking π_1, is a countably infinite dimensional $\mathbb{F}_p$-vector space.

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