Quantum Symmetry via K-theory

Bowen Yang, joint work with Mattie Ji

Harvard CMSA
Department of Mathematics, Harvard University

Algebraic Structures in Topology 2026

Joint work with Mattie Ji

Based on:

- *Quantum Cellular Automata: The Group, the Space, and the Spectrum*
  Mattie Ji and Bowen Yang, arXiv:2602.16572.
- *K-Theoretic Obstructions to Linearizing QCA Representations*
  Mattie Ji and Bowen Yang, arXiv:2606.19657.

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Quantum Spin Systems and Gapped Phases

Quantum spin system. A quantum spin system consists of local degrees of freedom placed on a lattice X, for example $X = \mathbb{Z}^d$.

- Observables form a C^*-algebra $\mathcal{A}(X)$.
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Beaudry: From Stabilizer Codes to Fracton Phases of Matter

Jun 15, 2026, 9:00 AM
1h 30m
Institut Henri Poincaré

Description

Fracton orders are quantum systems characterized by the presence of excitations with restricted mobility. Many examples are described by stabilizer codes, that is, lattice Hamiltonians constructed from commuting generalized Pauli matrices. In this lecture series, we will explore the algebraic framework for fracton phases and mutual statistics that emerges from the study of stabilizer codes.

The course will be mostly based on work of Haah and Ruba-Yang, as well as joint work with Hermele, Shirley, Wickenden and Yang.

Figure: Available on YouTube

Invertible Phases

Nevertheless, the invertible phases are expected to correspond to invertible TQFTs.

Theorem/Conjecture (Kitaev, Freed–Hopkins, Kapustin, Kubota...)

Invertible phases assemble into an Ω-spectrum

$$\mathbb{IP} = \{\mathrm{IP}(\mathbb{Z}^d)\}_{d\geq 0}, \qquad \pi_0\big(\mathrm{IP}(\mathbb{Z}^d)\big) \cong \mathcal{I}(\mathbb{Z}^d).$$

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Theorem 1.1. *There is a 1:1 correspondence*

$$\begin{Bmatrix} \text{deformation classes of reflection positive} \\ \text{invertible } n\text{-dimensional extended topological} \\ \text{field theories with symmetry group } H_n \end{Bmatrix} \cong [MTH, \Sigma^{n+1} I\mathbb{Z}(1)]_{\mathrm{tor}}. \tag{1.2}$$

Figure: Freed-Hopkins 2016

Quantum Cellular Automata (QCA)

Definition: A QCA in dimension d is a strictly locality-preserving automorphism

$$\alpha : \mathcal{A}(\mathbb{Z}^d) \to \mathcal{A}(\mathbb{Z}^d).$$

Structure[1]**:**

- QCA form a group $\mathcal{Q}^*(\mathbb{Z}^d)$ under composition
- Contains a normal subgroup $\mathcal{C}^*(\mathbb{Z}^d)$ of quantum circuits

$$\mathcal{Q}^*(\mathbb{Z}^d)/\mathcal{C}^*(\mathbb{Z}^d) \quad \text{is abelian (Freedman–Hastings).}$$

- A QCA representation is a homomorphism

$$\varphi : G \longrightarrow \mathcal{Q}^*(\mathbb{Z}^d).$$

[1] After adding arbitary ancilla to stabilize

QCA action on States

Given a state

$$\omega : \mathcal{A}(\mathbb{Z}^d) \to \mathbb{C},$$

a QCA α produces a new state

$$\omega \circ \alpha : \mathcal{A}(\mathbb{Z}^d) \to \mathcal{A}(\mathbb{Z}^d) \to \mathbb{C}.$$

This means a QCA representation defines a group action on the quantum system or a *quantum G-symmetry*.

Key fact:

- Circuits $\mathcal{C}^*(\mathbb{Z}^d)$ do not change phase

Hence we obtain a group homomorphism

$$\mathcal{Q}^*(\mathbb{Z}^d)/\mathcal{C}^*(\mathbb{Z}^d) \longrightarrow \mathcal{I}(\mathbb{Z}^d).$$

QCA probes invertible phases.

Stable Homotopy Theory

Theorem (Ji-Y.)

Using algebraic K-theory, for a nice enough metric space X, we construct an Ω-spectrum

$$\mathbb{QCA}(X) = \{\mathrm{QCA}(X \times \mathbb{Z}^d)\}_{d \geq 0}.$$

such that

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The Elephant in the Room

How is $\mathbb{QCA}$ related to $\mathbb{IP}$

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Nevertheless, the QCA spectrum has other important applications.

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Algebra of Observables

$$X = \mathbb{Z}$$

$$\cdots \quad \underset{-2}{\mathrm{M}_{q(-2)}(\mathbb{F})} \quad \underset{-1}{\mathrm{M}_{q(-1)}(\mathbb{F})} \quad \underset{0}{\mathrm{M}_{q(0)}(\mathbb{F})} \quad \underset{1}{\mathrm{M}_{q(1)}(\mathbb{F})} \quad \underset{2}{\mathrm{M}_{q(2)}(\mathbb{F})} \quad \cdots$$

More precisely,

$$\mathcal{A}(X,q) = \operatorname*{colim}_{\text{finite subsets } S \subseteq X} \underbrace{\bigotimes_{s \in S} M_{q(s)}(\mathbb{F})}_{=:\mathcal{A}(S,q)},$$

where for $S \subseteq T$ finite subsets,

$$\mathcal{A}(S,q) \to \mathcal{A}(T,q)$$

is given by $- \otimes \mathrm{id}_{\mathcal{A}(T-S,q)}$, tensoring with the identity operator on $T - S$.

Quantum Cellular Automata

A quantum cellular automation α is an algebra automorphism

$$\alpha : \mathcal{A}(X, q) \to \mathcal{A}(X, q)$$

of finite spread, meaning that there exists some $\ell > 0$ such that for all $\gamma \in M_{q(x)}(\mathbb{F})$, $\alpha(\gamma) \in \mathcal{A}(D_\ell(x), q)$.

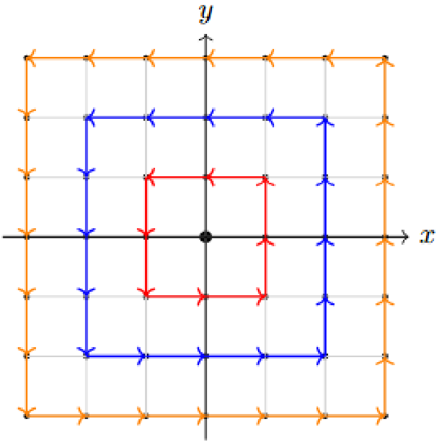
In other words, a QCA is an "automorphism that moves components away in a distance that is uniformly bounded".

Definition:

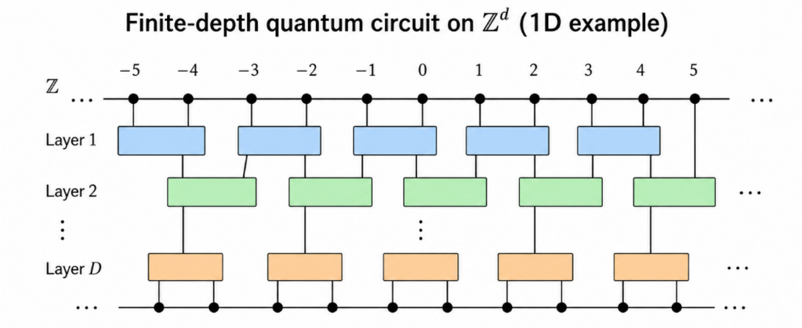
The group of QCAs over $\mathcal{A}(X, q)$ is denoted $\mathcal{Q}(X, q)$.

Examples

Annular shift on $\mathbb{Z}^2$ (2D example)



Examples



A special case: onsite QCA given by $\alpha \in \prod_{x \in X} PGL_{q_x}(\mathbb{F})$ or $\prod_{x \in X} PU(q_x)$.

Linearization of QCA representation

Given a QCA representation $\varphi : G \to \mathcal{Q}(X,q)$. Does there exist a QCA $\beta \in \mathcal{Q}(X,q)$ such that the following diagram can be completed?

$$
\begin{tikzcd}
 & \prod_{x\in X} \mathrm{GL}_{q_x}(\mathbb{F}) \arrow[d, two heads] \\
 & \prod_{x\in X} \mathrm{PGL}_{q_x}(\mathbb{F}) \arrow[d, hook] \\
G \arrow[uur, dashed] \arrow[ur, dashed] \arrow[r, "\beta\varphi\beta^{-1}"'] & \mathcal{Q}(X,q)
\end{tikzcd}
$$

Figure: Linearization

A Classical Result

Question (Linearization). Does there exist a lift $\tilde{\varphi}$ to the following diagram?

$$\begin{tikzcd} & GL(V) \arrow[d] \\ G \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"'] & PGL(V). \end{tikzcd}$$

This question has a well-known answer in terms of an obstruction class in group cohomology.

Theorem 1.1. *Let $\varphi : G \to PGL(V)$ be a projective representation over a field $\mathbb{F}$. Choose lifts $U_g \in GL(V)$ of $\varphi(g)$. Then there are scalars $\alpha(g,h) \in \mathbb{F}^\times$ such that*

$$U_g U_h = \alpha(g,h) U_{gh}.$$

The function α is a 2-cocycle, and its cohomology class

$$[\alpha] \in H^2(G, \mathbb{F}^\times)$$

is independent of the choice of lifts. The projective representation φ is linearizable if and only if $[\alpha] = 0$.

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Stable linearization and QCA spectrum

Stabilization

A QCA representation φ is *(trivially) stably linearizable* if $\varphi \otimes \psi_0 : G \to \mathcal{Q}(X, qr)$ admits a linearization where $\psi_0 : G \to \mathcal{Q}(X, r)$ is the trivial representation.

Theorem(Ji-Y.)

Let G be a finite or, more generally, rationally acyclic group. A QCA representation $\varphi : G \to \mathcal{Q}(X, q)$ defines a *cohomology class* $[\alpha] \in \mathbb{QCA}(X)^1(BG)$ which acts as an obstruction to stable linearization.

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Let's contrast it with the classical result.

The classical result

Let $\varphi : G \to PGL(V)$ be a projective representation over a field $\mathbb{F}$. The projective representation φ is linearizable only if the induced $[\alpha] \in H^2(G, \mathbb{F}^\times)$ vanishes.

Obstruction classes in ordinary cohomology

Theorem(Ji-Y.)

When G is finite (or more generally, rationally acyclic) and $X = \mathbb{Z}^n$, there are successive obstruction classes in

$$H^{i+1}(BG; \mathcal{Q}^*(\mathbb{Z}^{n-i})/\mathcal{C}^*(\mathbb{Z}^{n-i})),$$

for $0 \leq i \leq n$, and in

$$H^{n+2}(BG; U(1)).$$

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Proof (sketch): The obstruction class $[\alpha] \in \mathbb{QCA}(\mathbb{Z}^n)^1(BG)$ corresponds to a based map, up to homotopy,

$$BG \xrightarrow{\varphi^{\mathrm{st}}} B\mathcal{Q}(\mathbb{Z}^n) \xrightarrow{g} \mathbb{QCA}(\mathbb{Z}^{n+1}).$$

Construct the Postnikov tower of the homotopy fiber of g. The above classes are obstructions to lifting φ^{st} up this tower.