The kernel of the Birman–Craggs–Johnson homomorphism

Andrew Putman

University of Notre Dame

Algebraic Structures in Topology III
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Mapping class group

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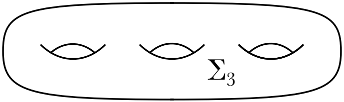
$\Sigma_g =$ cpt oriented genus g surface



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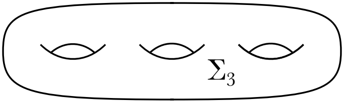

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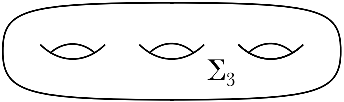

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Outline

(1) Basic facts about Mod_g.

(2) The Torelli group $\mathcal{I}_g$ and the Johnson homomorphism.

(3) 4-manifolds, Rochlin's theorem, and the Rochlin invariant (needed to understand torsion in H_1 of Torelli).

(4) The Birman–Craggs–Johnson BCJ homomorphism.

(5) New theorem giving generators for kernel of the BCJ homomorphism.

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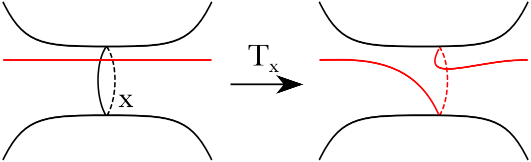

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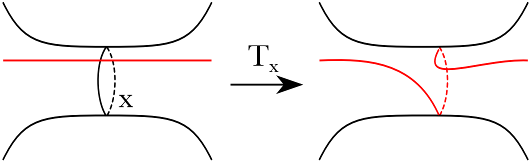
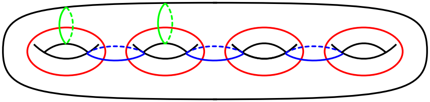


Theorem (Dehn 38 + Humphries 79)

Mod_g *is finitely generated by* $(2g+1)$ *Dehn twists:*



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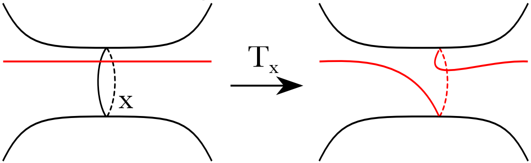

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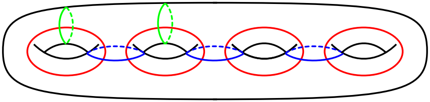

Theorem (McCool 75, or maybe Baily 60's)

Mod_g *is finitely presented.*

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Philosophy

We understand the group theory of $\mathrm{Sp}_{2g}(\mathbb{Z})$ very well since it is a group of matrices, so $\mathcal{I}_g$ is the mysterious seemingly nonlinear part of Mod_g.

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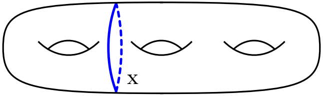

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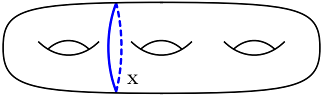
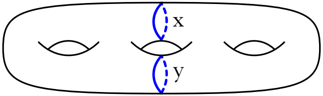


Bounding pair: $T_x T_y^{-1}$ w/ $x \cap y = \emptyset$ and $[x] = [y]$, i.e. $x \cup y$ bounds.



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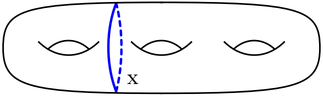

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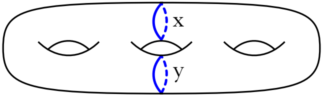

Theorem (Birman 71 + Powell 78)

$\mathcal{I}_g$ is gen. by sep twists and bounding pairs.

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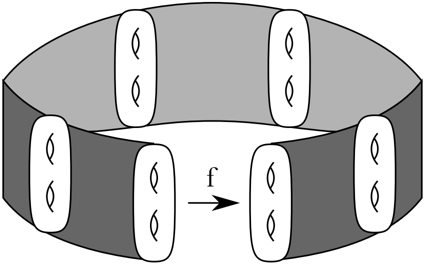

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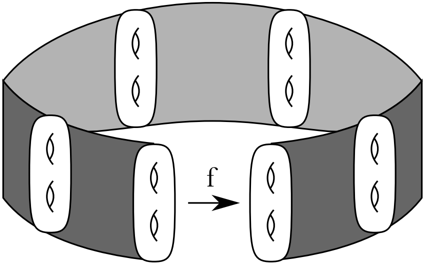

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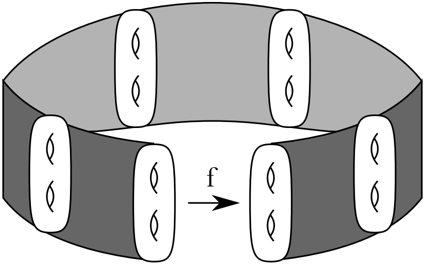

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$$\mathrm{H}^k(\mathrm{MapTor}(f)) \cong \mathrm{H}^k(\Sigma_g \times \mathbb{S}^1) \cong \begin{cases} \mathbb{Z} & \text{if } k = 0,3 \\ \mathbb{Z} \oplus \mathrm{H}^1(\Sigma_g) & \text{if } k = 1,2. \\ 0 & \text{otherwise.} \end{cases}$$

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Cup product structure on $\mathrm{H}^\bullet(\mathrm{MapTor}(f))$ determined by triple cup products of elements of $H \subset \mathrm{H}^1(\mathrm{MapTor}(f))$.

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Remark

Many alternate definitions of Johnson homomorphism (action on nilpotent quotients of π_1, monodromy of Jacobians, bordism, etc.)

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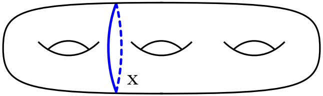

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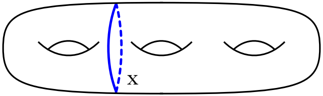

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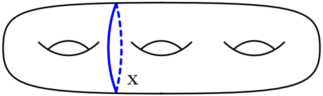

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Corollary

For $g \geq 3$, the abelianization $\mathrm{H}_1(\mathcal{I}_g)$ is $(\wedge^3 H)/H$ plus 2-torsion, where 2-torsion spanned by images of separating twists.

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Theorem (Ershov–He 17, Church–Ershov–Putman 17)

For $g \geq 4$, the Johnson kernel $\mathcal{K}_g$ is fin gen.

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Remark
If $\pi_1(N^4) = 1$, then N^4 is spin iff Q_N is even.

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Remark

Key to original proof is fact that the image of the stable J-homomorphism on π_3^s is $\mathbb{Z}/24$.

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- Van der Blij implies that $\sigma(N^4)$ divisible by 8.
- Want to define $\mathfrak{r}(M^3) = \sigma(N^4)/8 \pmod 2$. Must check independent of N^4.

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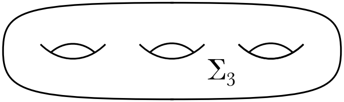

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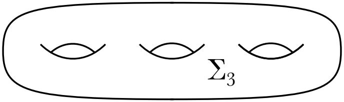

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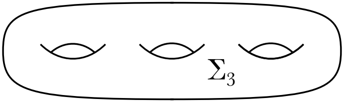

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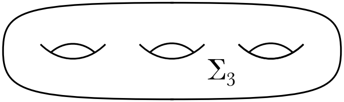

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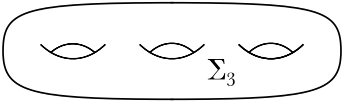

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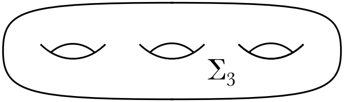

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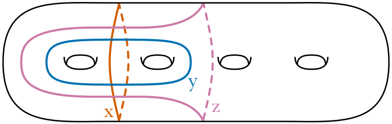

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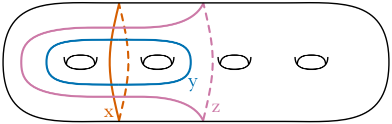
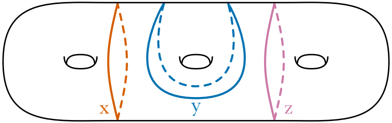


Tri-separating pants maps: products $T_x T_y T_z$ of three disjoint separating twists that cobound a 3-holed sphere ("pair of pants"):



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Corollary (Johnson)

For $g \geq 3$, the kernel of the BCJ homomorphism is $[\mathcal{I}_g, \mathcal{I}_g]_{\mathbb{F}_2}$, so it induces isomorphism $\mathrm{H}_1(\mathcal{I}_g; \mathbb{F}_2) \cong B_3^0$.